

FREE SEMIGROUPOID ALGEBRAS

DAVID W. KRIBS AND STEPHEN C. POWER

ABSTRACT. Every countable directed graph generates a Fock space Hilbert space and a family of partial isometries. These operators also arise from the left regular representations of free semigroupoids derived from directed graphs. We develop a structure theory for the weak operator topology closed algebras generated by these representations, which we call *free semigroupoid algebras*. We characterize semisimplicity in terms of the graph and show explicitly in the case of finite graphs how the Jacobson radical is determined. We provide a diverse collection of examples including; algebras with free behaviour, and examples which can be represented as matrix function algebras. We show how these algebras can be presented and decomposed in terms of amalgamated free products. We determine the commutant, consider invariant subspaces, obtain a Beurling theorem for them, conduct an eigenvalue analysis, give an elementary proof of reflexivity, and discuss hyper-reflexivity. Our main theorem shows the graph to be a complete unitary invariant for the algebra. This classification theorem makes use of an analysis of unitarily implemented automorphisms. We give a graph-theoretic description of when these algebras are *partly free*, in the sense that they contain a copy of a free semigroup algebra.

1. INTRODUCTION

We initiate the study of a new class of operator algebras which we call *free semigroupoid algebras*. These algebras include, as special cases, the space H^∞ realized as the *analytic Toeplitz algebra* [13, 16] and the prototypical *free semigroup algebras* \mathfrak{L}_n of Popescu, Davidson and Pitts (see [1, 8, 9, 10, 25, 30, 31]). However our context also embraces finite-dimensional operator algebras (inflation algebras), finite and infinite matrix function algebras, as well as operator algebras with free or partly free structure. Thus we have a unifying framework for the free

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semigroup algebras which includes the classical analytic Toeplitz algebra as part of a general class, rather than as an exceptional case. The framework also includes various nest algebras and finite dimensional digraph algebras but in a represented form wherein the commutant algebras have a similar character.

The generators of these algebras are families of partial isometries and projections which arise from countable directed graphs. Each graph G naturally determines a generalized Fock space Hilbert space and partial creation operators which act on the space. Alternatively, these operators come from the left regular representation of the *free semigroupoid* derived from the directed graph, thus giving further credence to the terminology.

The norm closed versions of these algebras, in the case of finite graphs, were considered by Muhly [28] with particular reference to commutant lifting, the Shilov boundary and the C^* -envelope. Subsequent work with Solel [29] on more general (norm-closed) tensor algebras addressed the structure of ideals, Wold decompositions and a Beurling type invariant subspace theorem. There is some overlap here with our development which we point out in Sections 4 and 8.

Presently there is considerable interest in C^* -algebras generated by families of partial isometries associated with directed graphs and how these algebras relate to the Elliott classification programme. The set of generators in this case is sometimes referred to as a Cuntz-Krieger E -family (for instance see [3, 26, 27]). On the other hand, our generators are of Cuntz-Krieger-Toeplitz type and we shall see that it is natural to include the initial projections. Moreover, our analysis is thoroughly non-selfadjoint and spatial: we consider one-sided generalized Fourier expansions, the Jacobson radical, invariant subspaces, spatially implemented automorphisms, reflexivity and Beurling type theorems. Much of this analysis goes into the proof of our main classification theorem, that the algebras are unitarily equivalent if and only if their directed graphs are isomorphic.

Note added in proof. This paper has motivated a number of recent papers on non-selfadjoint graph algebras and related topics including [14, 17, 18, 19, 20, 21, 22, 23, 33].

2. MAIN RESULTS

In this section we outline the main results of the paper. Every countable directed graph G generates in a recursive way a tree graph and an associated Hilbert space \mathcal{H}_G . On this Hilbert space, which can be viewed as a generalized Fock space, there are natural ‘partial creation

operators' $\{L_e\}$, one for each directed edge, and projections $\{L_x\}$, one for each vertex [28]. Alternatively, the family $\{L_e, L_x\}$ arises through the left regular representation λ_G of what we term the *free semigroupoid of G* determined by the directed paths and vertices in G . We develop a structure theory for the WOT-closed algebras \mathfrak{L}_G generated by families $\{L_e, L_x\}$, which we call *free semigroupoid algebras*.

There is also a natural right regular representation ρ_G determined by G . This yields partial isometries and projections $\{R_e, R_x\}$ on \mathcal{H}_G which commute with $\{L_e, L_x\}$. In fact, the WOT-closed algebra \mathfrak{R}_G generated by $\{R_e, R_x\}$ coincides with the commutant $\mathfrak{L}'_G = \mathfrak{R}_G$. Furthermore, $\mathfrak{R}'_G = \mathfrak{L}_G$ and hence \mathfrak{L}_G is its own second commutant. We establish this by first observing that \mathfrak{L}_G is unitarily equivalent to \mathfrak{R}_{G^t} , where G^t is the *transpose graph* with directed graph obtained from G simply by reversing the directions of all edges. In general, the algebras \mathfrak{R}_G and \mathfrak{R}_{G^t} can be different and so this is a departure from the free semigroup case, where \mathfrak{L}_n and \mathfrak{R}_n are unitarily equivalent just by symmetry. The methods here provide Fourier expansions for all elements of \mathfrak{L}_G , giving us a key technical device.

The question of semisimplicity is considered in Section 5 and we prove \mathfrak{L}_G is semisimple precisely when G is transitive in each component; that is, every edge lies on a cycle. Also we show for a finite graph G that the Jacobson radical of \mathfrak{L}_G is WOT-closed, nilpotent and equal to the ideal generated by the L_e which correspond to edges $e \in E(G)$ which do not lie on a cycle in G . Further, we show there is a block matrix decomposition of $\mathcal{B}(\mathcal{H}_G)$ such that the radical is the off-diagonal component of \mathfrak{L}_G in this decomposition.

In Section 6 we discuss several examples illustrating the diversity of the algebras \mathfrak{L}_G . In particular, a matrix function representation is obtained in the case of the graphs which are a single directed cycle. Also we indicate how amalgamations of graphs correspond to the (WOT-closed) free product with amalgamation of the associated free semigroupoid algebras.

We begin the analysis of the invariant subspace structure in Section 7. We first prove \mathfrak{L}_G is reflexive by considering some obvious invariant subspaces associated with \mathfrak{R}_G . Next the eigenvalues and eigenvectors for \mathfrak{L}_G^* are computed. In Section 8 we prove an invariant subspace theorem of Beurling type [5, 9, 29, 30]. Every invariant subspace of \mathfrak{L}_G is generated by a wandering subspace for the algebra, and is the orthogonal direct sum of cyclic subspaces which are minimal in the sense that the cyclic vector is supported by some projection L_x . Each of these minimal cyclic invariant subspaces is the range of a partial isometry in \mathfrak{R}_G , and the choice of partial isometry is unique up

to a scalar multiple. We then use the Beurling Theorem to derive an explicit characterization of partial isometries in \mathfrak{L}_G , and this yields an *inner-outer factorization* [13, 9, 16, 31] for elements of \mathfrak{L}_G .

We resolve the classification problem for these algebras in Section 9 by proving G to be a complete unitary invariant of \mathfrak{L}_G . In fact, there are simple dimension formulae through which one can calculate the graph structure from the algebra and the ideal \mathfrak{L}_G^0 generated by the L_e . However, this ideal is not necessarily preserved under unitary equivalence and to correct for this we analyze the unitarily implemented automorphisms of \mathfrak{L}_G . In particular, we show that these automorphisms act transitively on sets of eigenvectors for \mathfrak{L}_G^* .

In the final section we introduce the notion of a *partly free* WOT-closed operator algebra. Meaning that the algebra contains a copy of the free semigroup algebra \mathfrak{L}_2 . In spirit, this is a non-selfadjoint analogue of the requirement that a C^* -algebra contains a copy of the Cuntz algebra \mathcal{O}_2 . In the finite vertex case we prove the algebras \mathfrak{L}_G are partly free precisely when G contains a *double-cycle*. We also consider the stronger notion of a *unitally partly free* WOT-closed operator algebra. In this case there is a unital injection of \mathfrak{L}_2 into the algebra, and so it contains a pair of isometries with mutually orthogonal ranges. For the algebras \mathfrak{L}_G we prove this happens exactly when G satisfies a graph theoretic condition determined by its double-cycles. We conclude the paper by discussing hyper-reflexivity for \mathfrak{L}_G .

3. FREE SEMIGROUPOID ALGEBRAS

Let G be a finite or countably infinite directed graph with edge set $E(G)$ and vertex set $V(G)$. Let $\mathbb{F}^+(G)$ be the *free semigroupoid* determined by G . By this we mean that $\mathbb{F}^+(G)$ consists of the set of vertices x of G and the finite paths w of edges e in G , together with the natural operation of concatenation of allowable paths. (This is also called the *path space* of G .) We view each vertex as a degenerate path. Given a path w in $\mathbb{F}^+(G)$ we write $w = ywx$ when the initial and final vertices of w are, respectively, x and y . We shall use the *range* and *source* maps r and s to indicate this, so that $s(w) = x$ and $r(w) = y$. At times it will be convenient to use the *transition matrix* $A = (a_{yx})$ associated with G , where a_{yx} is the number of directed edges in G from vertex x to vertex y .

We note that the terminology ‘free’ is appropriate as all paths are admissible elements of $\mathbb{F}^+(G)$ and there are no relations amongst words, that is, there are no reducible products in $\mathbb{F}^+(G)$ other than trivial reductions involving units. Furthermore, there is clearly a natural

groupoid, an inverse semigroup in fact (which deserves to be called the free groupoid of G) which contains $\mathbb{F}^+(G)$ as a semigroupoid, just as in the free group case.

Let $\mathcal{H}_G = \ell^2(\mathbb{F}^+(G))$ be the Hilbert space with orthonormal basis $\{\xi_w : w \in \mathbb{F}^+(G)\}$ indexed by elements of $\mathbb{F}^+(G)$. For each edge $e \in E(G)$ and vertex $x \in V(G)$, we may define partial isometries and projections on \mathcal{H}_G by:

$$L_e \xi_w = \begin{cases} \xi_{ew} & \text{if } r(w) = s(e) \\ 0 & \text{if } r(w) \neq s(e) \end{cases}$$

and

$$L_x \xi_w = \begin{cases} \xi_{xw} = \xi_w & \text{if } r(w) = x \\ 0 & \text{if } r(w) \neq x \end{cases}$$

We shall use the convention $\xi_{ew} = 0$ if $r(w) \neq s(e)$. The family $\{L_e, L_x\}$ also arises [28] through the left regular representation $\lambda_G : \mathbb{F}^+(G) \rightarrow \mathcal{B}(\mathcal{H}_G)$, with $\lambda_G(e) = L_e$, and $\lambda_G(x) = L_x$.

Remark 3.1. From the C^* -algebra perspective, the partial isometries $\{L_e\}$ are of Cuntz-Krieger-Toeplitz type in the sense that the C^* -algebra generated by $\{L_e\}_{e \in E(G)}$ is generally the extension by the compact operators of a Cuntz-Krieger C^* -algebra [3, 26, 27].

As we see in later examples, there is a useful interpretation of the actions of the operators L_e (and the companion operators R_e) in terms of the natural tree graph whose nodes are labelled by the paths w . This tree is generated recursively from the ‘tree top graph’ consisting of the edges of G directed down from the vertices of G . The nodes w correspond to the basis vectors ξ_w and the operators L_e and R_e correspond to visualizable partial bijections of the tree structure. The tree perspective is also useful in the analysis of eigenvectors for \mathfrak{L}_G^* in Section 7.

Definition 3.2. The *free semigroupoid algebra* determined by G is the weak operator topology closed algebra generated by $\{L_e, L_x\}$,

$$\begin{aligned} \mathfrak{L}_G &= \text{WOT} - \text{Alg } \{L_e, L_x : e \in E(G), x \in V(G)\} \\ &= \text{WOT} - \text{Alg } \{\lambda_G(w) : w \in \mathbb{F}^+(G)\}. \end{aligned}$$

Finally we define \mathfrak{L}_G , up to unitary equivalence, in terms of generators and relations together with a spatial condition. This leads naturally into the topic of representation theory of \mathfrak{L}_G which we will not pursue in this paper. Nevertheless, we introduce the basic idea of a free partial isometry representation π of a directed graph G , which in turn gives rise to an operator algebra \mathfrak{L}_π which one might refer to as a free semigroupoid algebra. In the purely atomic case, which is

also characterized by a simple spatial condition, the algebras \mathfrak{L}_π are all naturally isomorphic to \mathfrak{L}_G .

Definition 3.3. Let G be a countable directed graph and let \mathcal{H} be a separable Hilbert space. A *free partial isometry representation* π of G is a pair of maps

$$\pi : V(G) \longrightarrow \text{Proj}(\mathcal{H}), \quad \pi : E(G) \longrightarrow \text{Pisom}(\mathcal{H}),$$

denoted $x \rightarrow P_x$ and $e \rightarrow S_e$ respectively, such that

- (i) given $s(e) = x$, the initial projection $S_e^* S_e = P_x \neq 0$,
- (ii) the projections P_x have pairwise orthogonal ranges and sum to the identity operator I ,
- (iii) the projections $\{S_e S_e^* : e \in E(G)\}$ are pairwise orthogonal and for each $x \in V(G)$,

$$E_x = P_x - \sum_{r(e)=x} S_e S_e^* \geq 0.$$

Such a partial isometry representation of G is indeed free in the sense that for each word w in $\mathbb{F}^+(G)$ the corresponding operator $\pi(w)$ (defined as the natural product) is a non-zero partial isometry and so we obtain a faithful representation of $\mathbb{F}^+(G)$. We write \mathfrak{L}_π for the weak operator topology closed algebra generated by $\{\pi(e), \pi(x) : e \in E(G), x \in V(G)\}$. Let us further define a *purely atomic* free partial isometry representation to be one for which each of the vacuum projections E_x is non-zero, and the set of final projections $\pi(w)E_x\pi(w)^*$, for x in $V(G)$ and $w \in \mathbb{F}^+(G)$, sum to the identity. Then we have the following proposition.

Proposition 3.4. *If π is a purely atomic free partial isometry representation of the directed graph G , then \mathfrak{L}_π is naturally isometrically isomorphic to \mathfrak{L}_G by an isomorphism which is WOT – WOT continuous. If the vacuum projections E_x , $x \in V(G)$, are each of rank one, then \mathfrak{L}_π and \mathfrak{L}_G are unitarily equivalent.*

Proof. If the vacuum projections each have rank one, then it follows that for a choice of unit vacuum vectors η_x with $E_x \eta_x = \eta_x$, the vectors $\eta_w = \pi(w) \eta_x$, for $wx = w$, make up an orthonormal basis. The correspondence $\eta_w \rightarrow \xi_w$ gives the desired unitary equivalence of \mathfrak{L}_π and \mathfrak{L}_G . If the vacuum projections are of infinite rank, then π is unitarily equivalent to the free partial isometry representation $\lambda_G^{(\infty)}$. Thus we have the natural isomorphisms $\mathfrak{L}_\pi \simeq \mathfrak{L}_\pi^{(\infty)} \simeq \mathfrak{L}_G^{(\infty)} \simeq \mathfrak{L}_G$. \blacksquare

4. COMMUTANT AND BASIC PROPERTIES

Let G be a directed graph, perhaps with a countable number of edges or vertices. Given $w = e_{i_k} \cdots e_{i_1} \in \mathbb{F}^+(G)$, let L_w be the partial isometry $L_w = L_{e_{i_k}} \cdots L_{e_{i_1}}$. We shall also use the notation $P_x \equiv L_x$ for vertex projections, or $P_i \equiv L_{x_i}$ if $V(G) = \{x_1, x_2, \dots\}$ has been enumerated. Observe that $L_w = P_{r(w)} L_w P_{s(w)}$. Define another family $\{R_e, R_x\}$ of partial isometries and projections on \mathcal{H}_G by the equations $R_e \xi_w = \xi_{we}$ and $R_x \xi_w = \xi_{wx}$. These operators also come from the right regular representation $\rho_G : \mathbb{F}^+(G) \rightarrow \mathcal{B}(\mathcal{H}_G)$, which yields partial isometries $\rho_G(w) \equiv R_{w'}$ for $w \in \mathbb{F}^+(G)$ acting on \mathcal{H}_G by the equations $R_{w'} \xi_v = \xi_{vw}$, where w' is the word w in reverse order. Observe that $R_{v'} L_w = L_w R_{v'}$ for all $v, w \in \mathbb{F}^+(G)$. In fact, in this section we show the algebra

$$\begin{aligned} \mathfrak{R}_G &= \text{WOT} - \text{Alg } \{R_e, R_x : e \in E(G), x \in V(G)\} \\ &= \text{WOT} - \text{Alg } \{\rho_G(w) : w \in \mathbb{F}^+(G)\} \end{aligned}$$

coincides with the commutant $\mathfrak{L}'_G = \mathfrak{R}_G$. We shall use notation $Q_x \equiv R_x$ for the vertex projections in \mathfrak{R}_G . Thus Q_x is the projection onto the subspace spanned by all ξ_w with $s(w) = x$. Let G^t denote the *transpose graph* of G . This is the directed graph obtained from G simply by reversing directions of all directed edges. If $v = e_{i_1} \cdots e_{i_k}$ is a product of edges in $E(G)$, then we let v^t be the product $v^t = e_{i_k}^t \cdots e_{i_1}^t$, where e^t is the directed edge in G^t which is e_i with direction reversed. Further define $x^t = x$ for $x \in V(G) = V(G^t)$.

Lemma 4.1. *The algebras \mathfrak{L}_G and \mathfrak{R}_{G^t} are unitarily equivalent via the map $W : \mathcal{H}_{G^t} \rightarrow \mathcal{H}_G$ defined by $W \xi_{v^t} = \xi_v$.*

Proof. The map W is easily seen to be a unitary operator. Given $v \in \mathbb{F}^+(G)$ and $e \in E(G)$,

$$(W^* L_e W) \xi_{v^t} = W^* L_e \xi_v = W^* \xi_{ev} = R_{e^t} \xi_{v^t}.$$

Hence $W^* L_e W = R_{e^t}$ for $e \in E(G)$, and similarly $W^* L_x W = R_{x^t}$ for $x \in V(G)$. Thus we have $W^* \mathfrak{L}_G W = \mathfrak{R}_{G^t}$. \blacksquare

As in the free semigroup case [9], we can consider the Cesaro operators associated with the partition $I = E_0 + E_1 + \dots$ where E_k is the projection onto $\text{span}\{\xi_w : |w| = k, w \in \mathbb{F}^+(G)\}$, the subspace spanned by basis vectors from paths of length k . These operators are given by

$$\Sigma_k(A) = \sum_{|j| < k} \left(1 - \frac{|j|}{k}\right) \Phi_j(A),$$

where the operators $\Phi_j(A) = \sum_{k \geq \max\{0, -j\}} E_k A E_{k+j}$ are the diagonals of A with respect to this block decomposition, and $\Sigma_k(A)$ converges to A in the strong operator topology for all $A \in \mathcal{B}(\mathcal{H}_G)$.

We mention that as a consequence of Lemma 4.1 one sees that the following result is a generalization of Proposition 5.4 from [29] where the commutant was shown to be determined by the algebra associated with the transpose graph.

Theorem 4.2. *The commutant of \mathfrak{R}_G coincides with \mathfrak{L}_G .*

Proof. We have observed above that \mathfrak{L}_G is contained in \mathfrak{R}'_G . To see the converse, fix $A \in \mathfrak{R}'_G$. We show that $A_x \equiv AP_x$ belongs to \mathfrak{L}_G for all $x \in V(G)$. Let $A\xi_x = Q_x A_x \xi_x = \sum_{s(w)=x} a_w \xi_w$. Define operators in \mathfrak{L}_G by

$$p_k(A_x) = \sum_{|w| < k; s(w)=x} \left(1 - \frac{|w|}{k}\right) a_w L_w.$$

We claim that $A_x = \text{SOT-lim}_{k \rightarrow \infty} p_k(A_x)$. This will be proved by showing that $p_k(A_x) = \Sigma_k(A_x)$. First observe that $\Phi_j(A_x) \in \mathfrak{R}'_G$ for all j . Indeed, this operator commutes with the R_e since A_x belongs to \mathfrak{R}'_G and $E_{k+1}R_e = R_e E_k$ for all k , while $\Phi_j(A_x)$ commutes with each Q_y since $Q_y E_k = E_k Q_y$ is the projection onto $\text{span}\{\xi_w : |w| = k, s(w) = y\}$ for all k . It follows that $\Sigma_k(A_x)$ belongs to \mathfrak{R}'_G for $k \geq 1$.

Now it is enough to show that $\Sigma_k(A_x)\xi_x = p_k(A_x)\xi_x$. If this is the case, then for $w = xw$ in $\mathbb{F}^+(G)$

$$\Sigma_k(A_x)\xi_w = R_w \Sigma_k(A_x)\xi_x = R_w p_k(A_x)\xi_x = p_k(A_x)\xi_w,$$

Whereas for $w = yw$ with $y \neq x$ we have

$$\Sigma_k(A_x)\xi_w = \Sigma_k(A_x)P_x P_y \xi_w = 0 = p_k(A_x)P_y \xi_w = p_k(A_x)\xi_w,$$

since P_x commutes with each E_l . However, observe that $\Phi_0(A_x)\xi_x = E_0(A_x)E_0\xi_x = a_x \xi_x$, and $\Phi_j(A_x)\xi_x = 0$ for $j > 0$. Further, for $j < 0$ we have

$$\Phi_j(A_x)\xi_x = (E_{-j}A_x)\xi_x = E_{-j} \sum_{s(w)=x} a_w \xi_w = \sum_{s(w)=x; |w|=-j} a_w \xi_w.$$

Hence it follows that

$$\Sigma_k(A_x)\xi_x = \sum_{|w| < k; s(w)=x} \left(1 - \frac{|w|}{k}\right) a_w \xi_w = p_k(A_x)\xi_x.$$

We have established that each $A_x = AP_x$ belongs to \mathfrak{L}_G . This completes the proof since $A = \sum_{x \in V(G)} AP_x$, the sum converging in the strong operator topology when $V(G)$ is infinite. \blacksquare

Remark 4.3. From the proof of this theorem elements of \mathfrak{L}_G can be seen to have Fourier expansions. In particular, if A belongs to \mathfrak{L}_G with $A\xi_x = AP_x\xi_x = Q_x(AP_x)\xi_x = \sum_{s(w)=x} a_w\xi_w$ for $x \in V(G)$, then $A\xi_v = R_vA\xi_x = \sum_{s(w)=x} a_w\xi_{wv}$ for $v = xv \in \mathbb{F}^+(G)$, and it follows that the Cesaro partial sums associated with the series $A \sim \sum_{w \in \mathbb{F}^+(G)} a_w L_w$ converge in the strong operator topology to A .

Corollary 4.4. *The commutant of \mathfrak{L}_G coincides with \mathfrak{R}_G .*

Proof. If W is the unitary from Lemma 4.1, we have

$$\mathfrak{R}'_{G^t} = (W^* \mathfrak{L}_G W)' = W^* \mathfrak{L}'_G W.$$

On the other hand, it also follows from the lemma and the definition of W that $\mathfrak{R}_G = W \mathfrak{L}'_{G^t} W^*$. But the theorem tells us $\mathfrak{L}'_{G^t} = \mathfrak{R}'_{G^t}$ and hence $\mathfrak{R}_G = W \mathfrak{R}'_{G^t} W^* = \mathfrak{L}'_G$. ■

The following are simple consequences of the previous two results.

Corollary 4.5. *Let G be a countable directed graph.*

- (i) \mathfrak{L}_G is its own second commutant, $\mathfrak{L}_G = \mathfrak{L}''_G$.
- (ii) \mathfrak{L}_G is inverse closed.

We can also describe the self-adjoint part of these algebras.

Corollary 4.6. *The set of normal elements in \mathfrak{L}_G is precisely the span of $\{P_x : x \in V(G)\}$. Similarly, the normal elements in \mathfrak{R}_G belong to the span of $\{Q_x : x \in V(G)\}$.*

Proof. The operators belonging to the span of the P_x are normal elements since the P_x are projections with pairwise orthogonal ranges. Let A be a normal element of \mathfrak{L}_G and put $\alpha_x = (A\xi_x, \xi_x)$ for $x \in V(G)$. Clearly ξ_x is an eigenvector for \mathfrak{L}_G^* , since it is an eigenvector for each of the generators, hence $A^*\xi_x = \overline{\alpha_x}\xi_x$ and by normality $A\xi_x = \alpha_x\xi_x$. However, recall that $\text{Ran}(P_x) = \text{span}\{\xi_w : r(w) = x\} = \text{span}\{R_w\xi_x : r(w) = x\}$. Thus, as A commutes with \mathfrak{R}_G we have

$$A\xi_w = R_wA\xi_x = \alpha_x\xi_w \quad \text{for } w = xw \in \mathbb{F}^+(G).$$

In other words, $AP_x = \alpha_x P_x$ and it follows that $A = \sum_{x \in V(G)} \alpha_x P_x$. ■

5. THE RADICAL

In this section we determine when \mathfrak{L}_G is semisimple and in the case of finite graphs we show how to compute the Jacobson radical of \mathfrak{L}_G strictly in terms of the graph structure. In particular, there is a block matrix decomposition of \mathfrak{L}_G in which the radical is the off-diagonal part.

A directed graph G is *transitive* if there are paths in both directions between every pair of vertices in G . A (*connected*) *component* of G is given by a maximal collection of vertices and edges which are joined in the undirected graph determined by G . By a *cycle*, we mean a path in G with the same initial and final vertices. It is not hard to see that G is transitive in each component precisely when every directed edge in G lies on a cycle. Let $B(G) \subseteq E(G)$ be the collection of edges $e \in E(G)$ which *do not* lie on a cycle. The set $B(G)$ is empty precisely when G is transitive in each component. The Jacobson radical is determined by these edges.

Theorem 5.1. *\mathfrak{L}_G is semisimple if and only if G is transitive in each component. When G has finitely many vertices, $|V(G)| = M < \infty$, the radical is nilpotent of degree at most M and is equal to the WOT-closed two-sided ideal generated by $\{L_e : e \in B(G)\}$.*

We begin by proving one of the implications in the theorem.

Lemma 5.2. *If G is transitive in each component, then \mathfrak{L}_G is semisimple. In particular, for every non-zero A in \mathfrak{L}_G , there is a path $w \in \mathbb{F}^+(G)$ such that $L_w A$ is not quasinilpotent.*

Proof. By Theorem 4.2, the Fourier expansion of $A \in \mathfrak{L}_G$ is determined by the vectors $\{A\xi_x : x \in V(G)\}$. If A is non-zero, then some $A\xi_x = \sum_{s(w)=x} a_w \xi_w$ is non-zero and there is a path v of minimal length such that $a_v \neq 0$. As G is transitive in each component, there is a path $u \in \mathbb{F}^+(G)$ such that uv is a cycle. Hence the paths $(uv)^k$ for $k \geq 1$ are cycles in $\mathbb{F}^+(G)$. Thus for $k \geq 1$, the expansion for $L_u A$ gives us

$$(L_u A)^k \xi_x = (L_u A)^{k-1} \sum_{s(w)=x} a_w \xi_{uw} = a_v^k \xi_{(uv)^k} + \sum_{w \neq (uv)^k} b_w \xi_w.$$

Thus for $k \geq 1$ we have

$$|(L_u A)^k|^{1/k} \geq |((L_u A)^k \xi_x, \xi_{(uv)^k})|^{1/k} = |a_v| > 0,$$

so the operator $L_u A$ has positive spectral radius and is not quasinilpotent. But recall the radical $\text{rad } \mathfrak{L}_G$ is equal to the largest quasinilpotent ideal in \mathfrak{L}_G . Thus $\text{rad } \mathfrak{L}_G = \{0\}$ when G is transitive in each component. \blacksquare

Towards the converse implication we obtain the following description of edges not lying on cycles.

Lemma 5.3. *The following assertions are equivalent for $e \in E(G)$.*

- (i) $L_e \in \text{rad } \mathfrak{L}_G$.
- (ii) $e \in B(G)$.

(iii) $(AL_e)^2 = 0$ for all $A \in \mathfrak{L}_G$.

Proof. By considering Fourier expansions, the last two conditions are easily seen to be equivalent to the requirement $L_w^2 = L_{w^2} = 0$ whenever $w \in \mathbb{F}^+(G)$ is a path which includes e as an edge. The characterization of $\text{rad } \mathfrak{L}_G$ used in the previous proof shows that (iii) \Rightarrow (i). Finally, if (ii) fails, then there is a path $u \in \mathbb{F}^+(G)$ such that $(ue)^k$ is a cycle for $k \geq 1$, and we may argue as in the previous proof to show that (i) fails. \blacksquare

Proof of Theorem 5.1. The algebra \mathfrak{L}_G is semisimple when G is transitive in each component by Lemma 5.2. On the other hand, if there is a component in G which is not transitive, then the set $B(G)$ is nonempty and Lemma 5.3 gives an edge $e \in B(G)$ with $L_e \in \text{rad } \mathfrak{L}_G$. Thus \mathfrak{L}_G has non-zero radical in this case. It remains to show the radical is nilpotent of degree at most M and equal to the WOT-closed ideal generated by $\{L_e : e \in B(G)\}$ when the cardinality of the vertex set $V(G)$ is $M < \infty$.

Let \mathcal{J} be the WOT-closed two-sided ideal in \mathfrak{L}_G generated by $\{L_e : e \in B(G)\}$. We first observe that the radical contains this ideal. For this we may use the following block matrix decomposition of \mathfrak{L}_G . We say that a subset H of edges and vertices of G is *maximally transitive* if: there are directed paths in both directions between every pair of vertices in H ; the initial and final vertices of every edge in H belong to H ; each edge between every pair of vertices in H also belongs to H ; and H is maximal with respect to these properties. Let $\{G_i\}_{i \in \mathcal{I}}$ be the maximally transitive components of G , and let $\{S_i\}_{i \in \mathcal{I}}$ be the projections $S_i = \sum_{x \in V(G_i)} P_x$. Then $I = (\sum_{i \in \mathcal{I}} \oplus S_i) \oplus (\sum_{x \notin \cup_{i \in \mathcal{I}} V(G_i)} \oplus P_x)$ and we may consider the block matrix form of \mathfrak{L}_G with respect to this spatial decomposition. By considering Fourier expansions for elements of \mathfrak{L}_G , it is not hard to see that the ideal \mathcal{J} is given by the off-diagonal entries of \mathfrak{L}_G in this decomposition. It follows that $\mathcal{J}^M = \{0\}$, and for all $X \in \mathfrak{L}_G$ and $A \in \mathcal{J}$ we have $(XA)^M = 0$. Hence \mathcal{J} is contained in $\text{rad } \mathfrak{L}_G$ and is nilpotent of degree at most M .

For the converse inclusion, suppose A belongs to $\text{rad } \mathfrak{L}_G$ with expansion scalars $\{a_w\}_{w \in \mathbb{F}^+(G)}$. We claim that a coefficient a_w is non-zero only if the path w includes an edge $e \in B(G)$. This will complete the proof, since the Cesaro sums for A would then belong to \mathcal{J} , and they converge in the strong operator topology to A . Suppose by way of contradiction that there is a path v with $a_v \neq 0$ which includes no edges from $B(G)$, and assume v is a path of minimal length with this property. Since every path in G which includes no edge from $B(G)$ must be part of some

cycle, we may choose a path $u \in \mathbb{F}^+(G)$ such that $uv = x(uv)x$ is a cycle in G . Hence $(uv)^k$ belongs to $\mathbb{F}^+(G)$ for $k \geq 1$. But it is clear that $(uv)^M$ is a path of minimal length amongst the paths in the expansion of $(L_u A)^M$ with non-zero coefficients. Further, the coefficient of $L_{(uv)^M}$ in this expansion is $(a_v)^M$. Thus we have $\|(L_u A)^{Mk}\|^{1/k} \geq |a_v|^M > 0$ for $k \geq 1$. Hence $(L_u A)^M = ((L_u A)^{M-1} L_u) A$ has positive spectral radius and is not quasinilpotent. This is a contradiction since A belongs to $\text{rad } \mathfrak{L}_G$, and the result follows. \blacksquare

Remark 5.4. From the block matrix decomposition of \mathfrak{L}_G used in the proof of Theorem 5.1 for the finite vertex case, we see the ideal $\text{rad } \mathfrak{L}_G$ is given by the WOT-closed ideal determined by the off-diagonal entries in this decomposition, which in turn is the WOT-closed two-sided ideal generated by $\{L_e : e \in B(G)\}$. This point is discussed further in the next section. We also mention that the general ideal structure of \mathfrak{L}_G has been characterized in [19].

6. EXAMPLES AND AMALGAMATED FREE PRODUCTS

We now examine several simple examples of graphs G and their operator algebras \mathfrak{L}_G . Some of these algebras admit natural representations as algebras of matrices or as subalgebras of (possibly infinite) matrix algebras over H^∞ . In this case some of their algebraic features such as the Jacobson radical and the structure of ideals become more apparent. The examples will also be used to illustrate the discussion at the end of this section where we consider the structure of \mathfrak{L}_G as an amalgamated free product.

Example 6.1. We first consider the single vertex cases. The algebra generated by the graph with a single vertex x and single loop edge $e = xex$ is unitarily equivalent to the classical analytic Toeplitz algebra H^∞ [13, 16]. Indeed, \mathcal{H}_G in this case may be naturally identified with the Hardy space H^2 , and under this identification L_e is easily seen to be unitarily equivalent to the unilateral shift U_+ .

The noncommutative analytic Toeplitz algebras \mathfrak{L}_n , $n \geq 2$ [1, 8, 9, 10, 25, 30, 31], the fundamental *free semigroup algebras*, arise from the graphs with a single vertex and n distinct loop edges. For instance, in the case $n = 2$ with loop edges $e = xex \neq f = xfx$, the space \mathcal{H}_G is identified with unrestricted 2-variable Fock space \mathcal{H}_2 . Under this identification, the operators L_e, L_f are unitarily equivalent to the natural creation operators on \mathcal{H}_2 which are the canonical Cuntz-Toeplitz isometries. Further, $P_x = I$, and thus $\mathfrak{L}_G \simeq \mathfrak{L}_2$.

Example 6.2. If G is a finite directed graph with no directed cycles, then the Fock space \mathcal{H}_G is finite dimensional and so too is \mathfrak{L}_G . For example, consider the graph with three vertices and two edges, labelled x_1, x_2, x_3, e, f where $e = x_2 e x_1, f = x_3 f x_1$. Then the Fock space is spanned by the vectors $\{\xi_{x_1}, \xi_{x_2}, \xi_{x_3}, \xi_e, \xi_f\}$ and a little reflection reveals that with this basis the general operator $\alpha L_{x_1} + \beta L_{x_2} + \gamma L_{x_3} + \lambda L_e + \mu L_f$ in \mathfrak{L}_G is represented by the matrix

$$\begin{bmatrix} \alpha & & & & \\ & \beta & & & \\ & & \gamma & & \\ \lambda & & & \beta & \\ \mu & & & & \gamma \end{bmatrix}$$

(unmarked entries are zero). Alternatively one can reorder the basis to view \mathfrak{L}_G as a subalgebra of $\mathcal{M}_5(\mathbb{C})$ consisting of matrices of the form

$$\begin{bmatrix} \alpha & 0 \\ \lambda & \beta \end{bmatrix} \oplus \begin{bmatrix} \alpha & 0 \\ \mu & \gamma \end{bmatrix} \oplus [\beta] \oplus [\gamma].$$

Algebraically, \mathfrak{L}_G is isometrically isomorphic to the so-called digraph algebra in $\mathcal{M}_3(\mathbb{C})$ consisting of the matrices

$$\begin{bmatrix} \alpha & 0 & 0 \\ \lambda & \beta & 0 \\ \mu & 0 & \gamma \end{bmatrix}.$$

Recall that a digraph algebra $\mathcal{A}(H)$ is a unital subalgebra of $\mathcal{M}_n(\mathbb{C})$ which is spanned by some of the standard matrix units of $\mathcal{M}_n(\mathbb{C})$. The graph H is transitive and reflexive and is such that the edges of H naturally label the relevant matrix units. If we take H to be the augmentation of our graph G by loops at the vertices, then we can view \mathfrak{L}_G as a faithful representation of $\mathcal{A}(H)$. Notably this representation is not a star extendible isomorphism since $\mathcal{A}(H)$ and \mathfrak{L}_G generate different C^* -algebras. In general, for a finite cycle-less graph G , \mathfrak{L}_G is isometrically isomorphic to $\mathcal{A}(\tilde{G})$ where \tilde{G} is the transitive completion of G with vertex loops added.

The commutant of \mathfrak{L}_G is best understood through Theorem 4.2. However, one can confirm directly that the commutant of \mathfrak{L}_G for this example consists of the matrices

$$\begin{bmatrix} a & & & & \\ & b & & & \\ & & c & & \\ & \lambda & & a & \\ & & \mu & & a \end{bmatrix}$$

with a, b, c, λ, μ in \mathbb{C} . These matrices in fact correspond to the operators $aR_{x_1} + bR_{x_2} + cR_{x_3} + \lambda R_e + \mu R_f$ which are the typical elements of \mathfrak{K}_G .

In the terminology of the second author [32] the operator algebras \mathfrak{L}_G are finitely acting, since they act on finite dimensional Hilbert spaces. (This is a stronger notion than finite dimensionality for an operator algebra.) Also the discussion above shows that the operator algebras \mathfrak{L}_G are the so-called inflation algebras of digraph algebras given in [32].

Example 6.3. For a simple matrix function algebra, we may consider the graph G with vertices x, y and edges $e = xex$, $f = yex$. Then \mathfrak{L}_G is generated by $\{L_e, L_f, P_x, P_y\}$. If we make the natural identifications $\mathcal{H}_G = P_x \mathcal{H}_G \oplus P_y \mathcal{H}_G \simeq H^2 \oplus H^2$ (respecting word length), then

$$L_e \simeq \begin{bmatrix} U_+ & 0 \\ 0 & 0 \end{bmatrix} \quad L_f \simeq \begin{bmatrix} 0 & 0 \\ U_+ & 0 \end{bmatrix} \quad P_x \simeq \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad P_y \simeq \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.$$

Thus, \mathfrak{L}_G is unitarily equivalent to

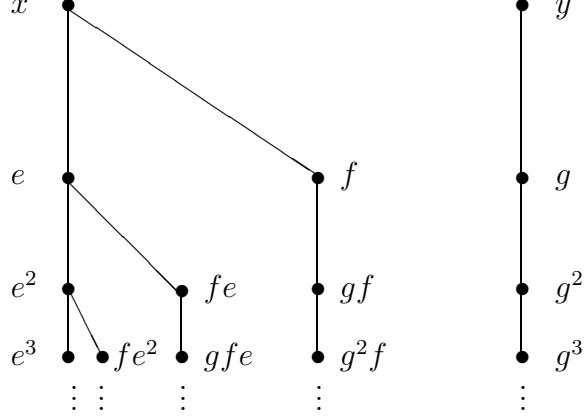
$$\mathfrak{L}_G \simeq \begin{bmatrix} H^\infty & 0 \\ H_0^\infty & \mathbb{C}I \end{bmatrix}$$

where H_0^∞ is the subalgebra of H^∞ functions h with $h(0) = 0$. With this representation of \mathfrak{L}_G , it is clear that $\text{rad } \mathfrak{L}_G$ is nilpotent of degree 2 and is given by the WOT-closed ideal generated by L_f . The commutant structure is less evident in this representation, nonetheless, we know it is generated by $\{R_e, R_f, Q_x, Q_y\}$ where $Q_y = \xi_y \otimes \xi_y^*$, $Q_x = I - Q_y$, $R_f = \xi_f \otimes \xi_y^*$ and $R_e = R_e P_x = P_x R_e$.

By simply adding a directed edge $g = xgy$ to the previous graph we obtain a very different algebra $\mathfrak{L}_{G'}$. Indeed, $\mathfrak{L}_{G'}$ is unitarily equivalent to its commutant $\mathfrak{L}'_{G'} = \mathfrak{K}_{G'} \simeq \mathfrak{L}_{(G')^t}$ since $(G')^t \simeq G'$. Furthermore, $\mathfrak{L}_{G'}$ is unittally partly free in the sense of Section 10 because it contains isometries with mutually orthogonal ranges, for instance, $U = L_e^2 + L_f L_g$ and $V = L_e L_g + L_g L_e$ are isometries which satisfy $U^* V = 0$. The algebra $\mathfrak{L}_{G'}$ will be discussed further in the context of amalgamated free products below.

Example 6.4. For an example of an infinite matrix function algebra, let G be the directed graph with transition matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and let $e = xex$, $f = yfx$ and $g = ygy$ be the directed edges of G . Notice that this graph is obtained from the graph G of the previous example by adding the edge g . Then \mathfrak{L}_G is generated by three partial isometries L_e, L_f, L_g and their initial projections $P_1 = L_e^* L_e = L_f^* L_f$ and $P_2 = L_g^* L_g$. These operators act on the Fock space \mathcal{H}_G with basis indexed by the

following tree:



Identify the Hardy space H^2 of the unit disc with each of the following ‘diagonal spaces’; $\mathcal{H}_1 = \mathcal{H}_e = \text{span} \{ \xi_x, \xi_e, \xi_{e^2}, \dots \}$,

$$\mathcal{H}_2 = L_f \mathcal{H}_e, \quad \mathcal{H}_3 = L_g L_f \mathcal{H}_e, \quad \dots, \quad \mathcal{H}_n = L_g^{n-2} L_f \mathcal{H}_e, \quad \dots$$

Also identify the space \mathcal{H}_g , similarly defined, with H^2 . With respect to the decomposition $\mathcal{H} = \left(\bigoplus_{n \geq 1} \mathcal{H}_n \right) \oplus \mathcal{H}_g$, the operator $\lambda L_e + \mu L_f + \nu L_g$ has block matrix form

$$\begin{bmatrix} \lambda T_z & 0 & & \\ \mu I & 0 & 0 & \\ & \nu I & 0 & 0 \\ & & \nu I & \ddots & \ddots \\ & & & \ddots & \ddots \end{bmatrix} \oplus [\nu T_z],$$

while $\alpha L_e^* L_e + \beta L_g^* L_g = \alpha P_x + \beta P_y$ has the form

$$\begin{bmatrix} \alpha I & & & \\ & \beta I & & \\ & & \beta I & \\ & & & \ddots \end{bmatrix} \oplus [\beta I].$$

It follows readily that \mathfrak{L}_G is naturally unitarily equivalent to the operator algebra of matrix functions

$$\begin{bmatrix} h_1(z) & & & \\ h_2(z) & \hat{h}(0) & & \\ h_3(z) & \hat{h}(1) & \hat{h}(0) & \\ h_4(z) & \hat{h}(2) & \hat{h}(1) & \hat{h}(0) \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix} \oplus [h(z)]$$

where $h_k \in H^\infty$ with $\sum_{k \geq 1} \|h_k\|^2$ finite, and where $h \in H^\infty$ with Fourier coefficients $\hat{h}(k)$.

With such an explicit matrix representation, one can examine the ideal structure and other algebraic aspects in a direct manner. It is clear, for example, that the Jacobson radical of \mathfrak{L}_G is given by the subspace for which $h_1 = h = 0$, and that the quotient by the radical is isomorphic to $H^\infty \oplus H^\infty$. Less evident is the structure of invariant subspaces for \mathfrak{L}_G which, we shall see, are generated by partial isometries in the commutant of \mathfrak{L}_G .

Example 6.5. The following algebras play a role in the analysis of Section 10. Consider the *cycle graph* C_n which has n vertices $\{x_1, \dots, x_n\}$ and n edges $\{e_1 = x_2 e_1 x_1, \dots, e_{n-1} = x_n e_{n-1} x_{n-1}, e_n = x_1 e_n x_n\}$. Let $\mathcal{H} \equiv \mathcal{H}_{C_n} = \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n$ be the decomposition of Fock space corresponding to the tree components (n infinite stalks in this case), so that $\mathcal{H}_i = Q_i \mathcal{H} \equiv Q_{x_i} \mathcal{H}$ for each i , and let $\mathcal{H}_{i,k} = L_{e_k}^* L_{e_k} \mathcal{H}_i = P_k Q_i \mathcal{H}$, for $1 \leq i, k \leq n$. Thus each subspace \mathcal{H}_i breaks up into the direct sum of n subspaces $\{\mathcal{H}_{i,k} : 1 \leq k \leq n\}$. With the natural (top down) basis ordering we have the identification of each space $\mathcal{H}_{i,k}$ with H^2 . With this identification note that the operator

$$L_{e_i} : \mathcal{H}_{i,k} \rightarrow \mathcal{H}_{i,k+1}$$

(with $\mathcal{H}_{i,n+1} = \mathcal{H}_{i,1}$) is the identity operator from H^2 to H^2 unless $k = i + 1 \pmod{n}$ in which case the operator is the unilateral shift T_z . It follows that \mathfrak{L}_{C_n} is realized as a subalgebra of the n -fold direct sum of matrix algebras $\mathcal{M}_n(H^\infty) \oplus \dots \oplus \mathcal{M}_n(H^\infty)$.

For example, when $n = 3$, with respect to the spatial decomposition

$$\mathcal{H} = \sum_{i=1}^3 \oplus Q_i \mathcal{H} = \left(\sum_{k=1}^3 \oplus \mathcal{H}_{1,k} \right) \oplus \left(\sum_{k=1}^3 \oplus \mathcal{H}_{2,k} \right) \oplus \left(\sum_{k=1}^3 \oplus \mathcal{H}_{3,k} \right)$$

the operator $\lambda L_{e_1} + \mu L_{e_2} + \nu L_{e_3}$ has operator matrix of the form

$$\begin{bmatrix} 0 & 0 & \nu T_z \\ \lambda I & 0 & 0 \\ 0 & \mu I & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & \nu I \\ \lambda T_z & 0 & 0 \\ 0 & \mu I & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & \nu I \\ \lambda I & 0 & 0 \\ 0 & \mu T_z & 0 \end{bmatrix}$$

while $\alpha L_{e_1}^* L_{e_1} + \beta L_{e_2}^* L_{e_2} + \gamma L_{e_3}^* L_{e_3} = \alpha P_1 + \beta P_2 + \gamma P_3$ has the form

$$\begin{bmatrix} \alpha I & 0 & 0 \\ 0 & \beta I & 0 \\ 0 & 0 & \gamma I \end{bmatrix} \oplus \begin{bmatrix} \alpha I & 0 & 0 \\ 0 & \beta I & 0 \\ 0 & 0 & \gamma I \end{bmatrix} \oplus \begin{bmatrix} \alpha I & 0 & 0 \\ 0 & \beta I & 0 \\ 0 & 0 & \gamma I \end{bmatrix}.$$

It follows that \mathfrak{L}_{C_3} can be identified with the matrix function algebra

$$\begin{bmatrix} h_{11} & zh_{12} & zh_{13} \\ h_{21} & h_{22} & zh_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \oplus \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ zh_{21} & h_{22} & zh_{23} \\ zh_{31} & h_{32} & h_{33} \end{bmatrix} \oplus \begin{bmatrix} h_{11} & zh_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ zh_{31} & zh_{32} & h_{33} \end{bmatrix}$$

where $h_{ij} \in H^\infty$ for $1 \leq i, j \leq 3$.

Once again, one can consider this explicit matrix function algebra representation in the analysis of the ideals of \mathfrak{L}_{C_n} . For example there is a unique maximal ideal, whose intersection with the centre of \mathfrak{L}_{C_n} identifies with zH^∞ , such that the quotient is not isomorphic to $\mathcal{M}_3(\mathbb{C})$. This exceptional quotient is isomorphic to the matrix algebra in $\mathcal{M}_3(\mathbb{C}) \oplus \mathcal{M}_3(\mathbb{C}) \oplus \mathcal{M}_3(\mathbb{C})$ consisting of the scalar matrices

$$\begin{bmatrix} a & 0 & 0 \\ d & b & 0 \\ f & e & c \end{bmatrix} \oplus \begin{bmatrix} a & h & i \\ 0 & b & 0 \\ 0 & e & c \end{bmatrix} \oplus \begin{bmatrix} a & 0 & i \\ d & b & j \\ 0 & 0 & c \end{bmatrix}.$$

Also one can verify readily that \mathfrak{L}_G is semisimple.

There is an alternative more succinct identification of the *cycle algebras* \mathfrak{L}_{C_n} which makes a connection with semicrossed product algebras. To see this identify $L_{x_i}\mathcal{H}$ with H^2 for each i in the natural way (respecting word length). Then $\mathcal{H} = L_{x_1}\mathcal{H} \oplus \dots \oplus L_{x_n}\mathcal{H} \simeq \mathbb{C}^n \otimes H^2$ and the operator $\alpha_1 L_{e_1} + \dots + \alpha_n L_{e_n}$ is identified with the operator matrix

$$\begin{bmatrix} 0 & & & & \alpha_n T_z \\ \alpha_1 T_z & 0 & & & \\ & \alpha_2 T_z & 0 & & \\ & & \ddots & \ddots & \\ & & & \alpha_{n-1} T_z & 0 \end{bmatrix}.$$

Writing $H^\infty(z^n)$ for the subalgebra of H^∞ arising from functions of the form $h(z^n)$ with h in H^∞ , the algebra \mathfrak{L}_{C_n} is readily identified with the matrix function algebra

$$\begin{bmatrix} H^\infty(z^n) & z^{n-1}H^\infty(z^n) & \dots & zH^\infty(z^n) \\ zH^\infty(z^n) & H^\infty(z^n) & & \vdots \\ \vdots & & \ddots & \\ z^{n-1}H^\infty(z^n) & \dots & & H^\infty(z^n) \end{bmatrix}.$$

If H^∞ is replaced with the disc algebra, this algebra becomes the matrix function realization of the semicrossed product $\mathbb{C}^n \times_\beta \mathbb{Z}_+$ for the cyclic shift automorphism of \mathbb{C}^n . See De Alba and Peters [12] for details. It follows that \mathfrak{L}_{C_n} is identifiable with the WOT-closed semicrossed product algebra $\mathbb{C}^n \times_\beta^\sigma \mathbb{Z}_+$.

Example 6.6. Let C_∞ be the infinite directed graph $C_\infty = (\mathbb{Z}, E)$ where the edge set $E = \{(n, n+1) : n \in \mathbb{Z}\}$, and hence the tree components for C_∞ give the two-way infinite graph generalization of the previous example. We have the decomposition of Fock space $\mathcal{H} \equiv \mathcal{H}_{C_\infty}$ into a direct sum of spaces \mathcal{H}_k , $k \in \mathbb{Z}$, where $\mathcal{H}_k = L_{e_k}^* L_{e_k} \mathcal{H} = P_k \mathcal{H}$ is naturally identified with H^2 . Pictorially, the basis elements of \mathcal{H}_k are diagonally distributed to the south east of vertex x_k . With respect to the decomposition $\mathcal{H} = \sum_{k \in \mathbb{Z}} \oplus \mathcal{H}_k = \ell^2(\mathbb{Z}) \otimes H^2$, the partial isometry L_{e_k} has the representation $e_{k+1,k} \otimes T_z$, where $\{e_{ij} : i, j \in \mathbb{Z}\}$ is the standard matrix unit system for the standard basis of $\ell^2(\mathbb{Z})$. Since $L_{e_k}^* L_{e_k}$ has the form $e_{k,k} \otimes I$ it follows that \mathfrak{L}_{C_∞} is unitarily equivalent to the operator algebra of matrices

$$(a_{ij} T_z^{j-i})_{i,j=-\infty}^\infty$$

where (a_{ij}) is the standard matrix of an operator in $\mathcal{T}_\mathbb{Z}$, the nest algebra on $\ell^2(\mathbb{Z})$ for the nest subspaces $\text{span}\{e_j : j \leq k\}$, $k \in \mathbb{Z}$. Once again the algebraic structure of \mathfrak{L}_{C_∞} becomes clear in this representation since \mathfrak{L}_{C_∞} is isomorphic to $\mathcal{T}_\mathbb{Z}$ by a WOT-WOT-continuous isometric isomorphism. In particular, the Jacobson radical of \mathfrak{L}_{C_∞} is determined by Ringrose's criterion [11] and the radical is not closed in the weak operator topology.

On the other hand, as before, it is less clear in this representation how to describe the commutant of \mathfrak{L}_{C_∞} and other spatial structure of the algebra. Our earlier Fock space arguments show that the commutant is unitarily equivalent to \mathfrak{L}_G where G is the transpose of C_∞ , and so, as with the cycle algebras \mathfrak{L}_{C_n} , the algebra \mathfrak{L}_{C_∞} is isomorphic to its commutant.

6.1. Amalgamated Free Products. We now describe how free semi-groupoid algebras may be presented and decomposed in terms of free products and amalgamated free products. The discussion here is independent of the rest of the paper. However, the gauge automorphisms of \mathfrak{L}_G are of interest in their own right and in the case of \mathfrak{L}_n we use the gauge automorphisms in the proof of Theorem 7.1.

Let G be a countable directed graph. It is convenient now to identify the Fock space elements $\xi_{e_{i_1} \dots e_{i_k}}$ with tensor products $\eta_{e_{i_1}} \otimes \dots \otimes \eta_{e_{i_k}}$ where $\{\eta_{e_{i_1}}, \dots, \eta_{e_{i_n}}\}$ is a fixed orthonormal basis for \mathbb{C}^n . This allows us to define certain automorphisms of \mathfrak{L}_G in a succinct manner in terms of their action on these admissible tensors. Thus, suppose that for each vertex pair x_i, x_j we have a complex unitary matrix $U_{i,j}$ of size $a_{ij} \times a_{ij}$ where, as before, a_{ij} is the number of edges in G from x_j to x_i . We

view $U_{i,j}$ as a unitary on

$$\mathbb{C}^{a_{ij}} = \text{span}\{\eta_e : e = x_i e x_j\}.$$

Now we may define a unitary $U = \sum \oplus U_{ij}$ on \mathbb{C}^n where $U\eta_e = U_{ij}\eta_e$ if $e = x_i e x_j$. For convenience write U_e for U_{ij} for any edge (there may be several) with $e = x_i e x_j$. By linear extension there is an associated operator \tilde{U} on the Fock space \mathcal{H}_G for G which fixes vacuum vectors such that

$$\tilde{U}\eta = (U_{e_1}\eta_{e_1}) \otimes \dots \otimes (U_{e_k}\eta_{e_k})$$

for all admissible tensors $\eta_{e_1} \otimes \dots \otimes \eta_{e_k}$. Note, of course, that in the expansion of the tensor $\tilde{U}\eta$ one obtains a linear combination of admissible tensors and hence \tilde{U} is well defined as a unitary on \mathcal{H}_G .

In the free semigroup case the unitaries \tilde{U} correspond to the so-called gauge unitaries in the formulation of quantum mechanics. The new aspect here is that gauge unitaries are also available for non-loop edges. Plainly the gauge unitaries \tilde{U} respect the natural grading of Fock space. Moreover, one can verify, by considering actions on the generators L_e , that the map Θ_U on \mathcal{L}_G defined by $\Theta_U(A) = \tilde{U}^* A \tilde{U}$ yields an injective endomorphism of \mathcal{L}_G . Since U is unitary this endomorphism is in fact an automorphism which we refer to as a *gauge automorphism* of \mathcal{L}_G .

The tensor presentation of Fock space above is a simple variation of the well known presentation of Fock space for the free semigroup algebra \mathcal{L}_n in the form $\mathcal{H}_n = \mathbb{C}\xi \oplus (\bigoplus_{N \geq 1} \mathcal{K}^{\otimes N})$, where \mathcal{K} is a Hilbert space with basis $\{\eta_{e_{i_1}}, \dots, \eta_{e_{i_n}}\}$ indexed by the loop edges for the single vertex graph for \mathcal{L}_n . In general we can write the Fock space for G as

$$\mathcal{H}_G = \left(\sum_{x \in V(G)} \oplus \mathbb{C}\xi_x \right) \oplus \left(\bigoplus_{N \geq 1} \mathcal{K}^{(N)} \right)$$

where $\mathcal{K}^{(N)}$ is spanned by the admissible tensors of length N .

On the other hand, \mathcal{H}_n can also be viewed as an n -fold free product of the Hilbert space of functions H^2 , with distinguished vector $\mathbf{1}$. In fact, in general, if (\mathcal{H}_i, ξ_i) is a family of Hilbert spaces with distinguished unit vectors, then their Hilbert space free product (\mathcal{H}, ξ) , denoted $*_{i \in \mathcal{I}}(\mathcal{H}_i, \xi_i)$, has the form

$$\mathcal{H} = \mathbb{C}\xi \oplus \bigoplus_{N \geq 1} \left(\bigoplus_{i_1 \neq i_2 \neq \dots \neq i_N} \overset{\circ}{\mathcal{H}}_{i_1} \otimes \dots \otimes \overset{\circ}{\mathcal{H}}_{i_N} \right)$$

where $\overset{\circ}{\mathcal{H}}_i = \mathcal{H} \oplus \mathbb{C}\xi_i$. Such a free product Hilbert space allows for the construction of a faithful representation $\pi = *_{i \in \mathcal{I}} \pi_i : *_{i \in \mathcal{I}} \mathcal{A}_i \rightarrow \mathcal{B}(\mathcal{H})$ of an (algebraic) free product of unital operator algebras \mathcal{A}_i ,

each represented on a Hilbert space \mathcal{H}_i by a faithful representation π_i . Viewing the operator algebras \mathcal{A}_i as represented on \mathcal{H}_i we suppress the notation π_i and define $*_{i \in \mathcal{I}} \mathcal{A}_i$ in the category of WOT-closed operator algebras to be the WOT-closure of the algebraic free product in the representation π . For more details see [15]. In this way we can define the WOT-closed algebra $*_{i \in \{1, \dots, n\}} H^\infty$, where H^∞ is represented on H^2 with distinguished vector $\mathbf{1}$, and this can be shown to coincide with \mathfrak{L}_n . However, in the special case of free products of the operator algebras \mathfrak{L}_n one may take a slightly more explicit approach: Let $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 = \mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2}$ be a decomposition associated with a partition of the basis $\{\eta_{e_{i_1}}, \dots, \eta_{e_{i_n}}\}$, where $n = n_1 + n_2$. Then \mathfrak{L}_n acts on the unrestricted Fock space \mathcal{H}_n , which can be recognized as the free product $(\mathcal{H}_{n_1}, \xi) * (\mathcal{H}_{n_2}, \xi)$ defined above. (This is a special case of (1) below.) Plainly \mathfrak{L}_{n_1} and \mathfrak{L}_{n_2} are naturally represented on this space by viewing each L_e as an appropriate creation operator. One can check that this representation agrees with our general free product representation π , and it follows that $\mathfrak{L}_{n_1} * \mathfrak{L}_{n_2}$ is unitarily equivalent to \mathfrak{L}_n .

Suppose now that G_1 and G_2 are finite directed graphs with a single identified vertex x say, and let $G = G_1 \sqcup_x G_2$ be the amalgamated graph. We want to identify \mathfrak{L}_G with a WOT-closed amalgamated free product $\mathfrak{L}_{G_1} *_{\mathbb{C}P_x} \mathfrak{L}_{G_2}$, where $\mathbb{C}P_x$ is the common subalgebra. Rather than placing this construction in a general context, let us take advantage of the explicit nature of the algebras \mathfrak{L}_G , as defined by creation operators, to give a direct definition of $\mathfrak{L}_{G_1} *_{\mathbb{C}P_x} \mathfrak{L}_{G_2}$. This definition, as before, relies on specifying an appropriate Hilbert space on which to represent \mathfrak{L}_{G_1} and \mathfrak{L}_{G_2} . The construction is similar to the free semigroup case except that only admissible tensors corresponding to finite paths are considered. Thus we have

$$\mathcal{H}_{G_1} = \text{span}\{\xi_{x_i}\} \oplus \bigoplus_{N \geq 1} \mathcal{K}_1^{(N)} \quad \text{and} \quad \mathcal{H}_{G_2} = \text{span}\{\xi_{y_j}\} \oplus \bigoplus_{N \geq 1} \mathcal{K}_2^{(N)}.$$

With $x_1 = y_1 = x$ understood we write $\mathcal{K}_1^{(N)} \otimes_x \mathcal{K}_2^{(M)}$ to denote the Hilbert space spanned by basis elements

$$(\eta_{e_{i_1}} \otimes \dots \otimes \eta_{e_{i_N}}) \otimes (\eta_{f_{j_1}} \otimes \dots \otimes \eta_{f_{j_M}})$$

where $e_{i_N} = e_{i_N}x$ and $f_{j_1} = xf_{j_1}$. Now we may define

$$(1) \quad \mathcal{H} = \mathcal{H}_{G_1} *_x \mathcal{H}_{G_2} \equiv \text{span}\{\xi_{x_i}, \xi_{y_j}\} \oplus \bigoplus_{N \geq 1}$$

$$\left(\bigoplus_{i_1 \neq i_2 \neq \dots \neq i_N} \left(\left[\bigoplus_{p_1 \geq 1} \mathcal{K}_{i_1}^{(p_1)} \right] \otimes_x \dots \otimes_x \left[\bigoplus_{p_N \geq 1} \mathcal{K}_{i_N}^{(p_N)} \right] \right) \right).$$

There are natural representations $\pi_i : \mathfrak{L}_{G_i} \rightarrow \mathcal{B}(\mathcal{H})$, which agree on the common projection P_x and we define $\mathfrak{L}_{G_1} *_{\mathbb{C}P_x} \mathfrak{L}_{G_2}$ to be the generated WOT-closed operator algebra. By simple manipulations as in the free semigroup case, we can identify this operator algebra with \mathfrak{L}_G . Thus we have the following.

Theorem 6.7. *Let G_1 and G_2 be finite directed graphs with a single identified vertex x and let $G = G_1 \sqcup_x G_2$ be the amalgamated graph. Then \mathfrak{L}_G is naturally unitarily equivalent to the amalgamated free product algebra $\mathfrak{L}_{G_1} *_{\mathbb{C}P_x} \mathfrak{L}_{G_2}$.*

One can also view $\mathfrak{L}_{G_1} *_{\mathbb{C}P_x} \mathfrak{L}_{G_2}$ as arising from the amalgamation over x of the left regular representations $\lambda_{G_1}, \lambda_{G_2}$ of the free semigroupoids $\mathbb{F}^+(G_1), \mathbb{F}^+(G_2)$. That is, from $\lambda_{G_1} *_x \lambda_{G_2}$, appropriately defined, where x is an identified unit of G_1 and G_2 . As in the group case (see [15]) $\lambda_{G_1} *_x \lambda_{G_2}$ identifies naturally with $\lambda_{G_1 \sqcup_x G_2} = \lambda_G$. We can now revisit the $\mathfrak{L}_{G'}$ of Example 6.3 and see that it is unitarily equivalent to $H^\infty *_{\mathbb{C}P_x} \mathfrak{L}_{C_2}$. More generally, if G is the graph formed by joining C_n and C_m at a single vertex x then

$$\mathfrak{L}_G \simeq (\mathbb{C}^n \times_\beta^\sigma \mathbb{Z}_+) *_{\mathbb{C}P_x} (\mathbb{C}^m \times_\beta^\sigma \mathbb{Z}_+).$$

7. REFLEXIVITY AND EIGENVECTORS

We first give an elementary proof that the algebras \mathfrak{L}_G are reflexive. Recall that given an operator algebra \mathfrak{A} and a collection of subspaces \mathcal{L} , the subspace lattice $\text{Lat } \mathfrak{A}$ consists of those subspaces left invariant by every member of \mathfrak{A} , and the algebra $\text{Alg } \mathcal{L}$ consists of all operators which leave every subspace of \mathcal{L} invariant. Every algebra satisfies $\mathfrak{A} \subseteq \text{Alg Lat } \mathfrak{A}$ and an algebra is *reflexive* if $\mathfrak{A} = \text{Alg Lat } \mathfrak{A}$.

Theorem 7.1. *\mathfrak{L}_G is reflexive.*

Proof. Let $A \in \text{Alg Lat}(\mathfrak{L}_G)$. Then $A(\overline{R\mathcal{H}_G}) \subseteq \overline{R\mathcal{H}_G}$ for all $R \in \mathfrak{R}_G$ since $\mathfrak{L}_G = \mathfrak{R}'_G$. Suppose that $A = AP_x$ for some $x \in V(G)$. For $u = yu \in \mathbb{F}^+(G)$, the subspace $R_u \mathcal{H}_G$ is the set of vectors in \mathcal{H}_G of the form $\sum_{s(w)=y} \beta_w \xi_{wu}$. Hence $A\xi_u = AR_u \xi_y = AR_u P_x \xi_y = 0$ when $x \neq y$, and otherwise $A\xi_u$ has expansion $\sum_{s(w)=x} \alpha_w^u \xi_{wu}$ when $r(u) = x$. We claim that the scalars α_w^u are independent of u ; that is, if $r(u) = x = r(v)$, then $\alpha_w^u = \alpha_w^v$ for all $w = wx$.

Assume the claim holds for the moment. Then there are scalars $\{\alpha_w\}_{w \in \mathbb{F}^+(G)}$, where $\alpha_w = 0$ for $s(w) \neq x$, with $A\xi_u = \sum_{s(w)=x} \alpha_w \xi_{wu}$ for all $u = xu \in \mathbb{F}^+(G)$ and $A\xi_u = 0$ otherwise. For such an operator, it is easy to check that A belongs to $\mathfrak{R}'_G = \mathfrak{L}_G$. In the general case, given $A \in \text{Alg Lat}(\mathfrak{L}_G)$ we may write $A = \sum_{x \in V(G)} AP_x$, the sum converging

in the strong operator topology for the infinite vertex case. Then the above argument can be applied to show each AP_x belongs to $\mathfrak{K}'_G = \mathfrak{L}_G$, and hence A is in \mathfrak{L}_G as required. Thus, we will prove the theorem by verifying the claim.

There are two cases to consider. For the first case let us suppose that there is an edge v with $v = xvy$ and $y \neq x$. Then the range \mathcal{M} of $R_x + R_v$ is spanned by the set of vectors $\{\xi_{wx} + \xi_{wv} : s(w) = x\}$ and the vectors in this set are pairwise orthogonal. Since $\mathcal{M} \in \text{Lat } \mathfrak{L}_G$ it follows that

$$A(\xi_x + \xi_v) = A(R_x + R_v)\xi_x = \sum_{s(w)=x} h_w(\xi_{wx} + \xi_{wv})$$

for some choice of scalars h_w . But $A\xi_x$ and $A\xi_v$ are given in terms of the coefficients α_w^x, α_w^v respectively and we conclude that $\alpha_w^x = h_w = \alpha_w^v$ for all w .

For the remaining case we have $u = xu$ only if $u = xux$ and we see that the compression algebra $P_x \mathfrak{L}_G|_{P_x \mathcal{H}_G}$ is unitarily equivalent to \mathfrak{L}_H where H is a single vertex subgraph of G . Thus \mathfrak{L}_H is unitarily equivalent to \mathfrak{L}_n for some $n \geq 1$ or \mathfrak{L}_H is unitarily equivalent to \mathbb{C} , and these algebras are known to be reflexive. We may assume then that $H \neq G$ and also that $\alpha_w^v = \alpha_w^x$ for all words v, w with $r(v) = x = r(w)$. Suppose that $w' = yw'x$ with $y \neq x$. It remains to show that $\alpha_{w'}^v = \alpha_{w'}^x$.

Suppose first that w' is not of the form w_1h where h is a path in H with $|h| \geq 1$. Consider the restriction operator $A_{w'} = L_{w'}^* A|_{\mathcal{H}_H}$. We show that $A_{w'}$ is in \mathfrak{L}_H . To this end, let $\mathcal{M} \in \text{Lat } \mathfrak{L}_H$ and define $\widetilde{\mathcal{M}} = \bigvee_{s(w)=x} L_w \mathcal{M}$ in $\text{Lat } \mathfrak{L}_G$. Then

$$A_{w'} \mathcal{M} = L_{w'}^* A \mathcal{M} \subseteq L_{w'}^* \widetilde{\mathcal{M}}.$$

In view of our assumption on w' we have that $L_{w'}^* L_w$ is non-zero only if $w = w'h$ with h a path in H . Thus

$$L_{w'}^* \widetilde{\mathcal{M}} \subseteq \bigvee_{w=xwx} L_w \mathcal{M} \subseteq \mathcal{M}.$$

We have shown that $A_{w'} \in \text{Alg Lat } \mathfrak{L}_H$ and so $A_{w'} \in \mathfrak{L}_H$. Hence there are scalars α_h such that

$$A_{w'} \xi_v = \left(\sum_{h=xhx} \alpha_h L_h \right) \xi_v$$

for all paths v in H . Thus we have

$$\begin{aligned}
 \sum_h \alpha_h \xi_{w'hv} &= L_{w'}(A_{w'} \xi_v) \\
 &= L_{w'} L_{w'}^* A \xi_v \\
 &= L_{w'} L_{w'}^* \left(\sum_w \alpha_w^v \xi_{wv} \right) \\
 &= \sum_h \alpha_{w'h}^v \xi_{w'hv}.
 \end{aligned}$$

This shows that $\alpha_{w'h}^v = \alpha_h$ for all $h = xhx$. In particular, we obtain $\alpha_w^v = \alpha_w^x$ for all words w with $w = ywx$, $y \neq x$, as desired. \blacksquare

We next compute the eigenvalues for \mathfrak{L}_G^* , by which we mean, with modest abuse of terminology, the values $\lambda = (\lambda_e)_{e \in E(G)}$ in \mathbb{C}^n , where n is the cardinality of $E(G)$, for which there is a unit vector ξ in \mathcal{H} such that $L_e^* \xi = \bar{\lambda}_e \xi$ for all $e \in E(G)$. Since the L_e are partial isometries with pairwise orthogonal final projections, we have

$$\sum_{e \in E(G)} |\lambda_e|^2 = \sum_{e \in E(G)} (L_e L_e^* \xi, \xi) \leq \|\xi\|^2 = 1.$$

In the free semigroup case, the open unit n -ball

$$\mathbb{B}_n = \left\{ \lambda = (\lambda_e)_{e \in E(G)} : \sum_{e \in E(G)} |\lambda_e|^2 < 1 \right\}$$

forms the set of eigenvalues for \mathfrak{L}_n^* $[1, 9]$. In general the eigenvalues for \mathfrak{L}_G^* form a proper subset of the unit n -ball, with structure determined by lower dimensional unit balls.

The set of eigenvalues for \mathfrak{L}_G^* will be described explicitly in terms of G . We begin by pointing out a special case which is quite different from the semigroup case.

Proposition 7.2. *If G has no loop edges, $a_{xx} = 0$ for all x in $V(G)$, then $\lambda = \vec{0} \in \mathbb{C}^n$ is the only eigenvalue for \mathfrak{L}_G^* .*

Proof. In this case $L_e^2 = L_{e^2} = 0$ for all $e \in E(G)$, as no words of the form $e^2 w$ belong to $\mathbb{F}^+(G)$. As every eigenvalue $\lambda = (\lambda_e)_{e \in E(G)} \in \mathbb{C}^n$ for \mathfrak{L}_G^* is determined by equations $L_e^* \xi = \bar{\lambda}_e \xi$, it follows that $\lambda = \vec{0}$. \blacksquare

We require some extra notation to state the following theorem. We shall say the ‘tree top graph’ associated with a directed graph G (the top two levels of the tree components for \mathcal{H}_G) has a *standard ordering* if the *saturation* G_x (the set of all paths which start at x) at every vertex x in $V(G)$ has all its edges which finish at x lying to the left of all other

edges in G_x . Also, we shall use the notation $0_k = \vec{0} \in \mathbb{C}^k$. Further, we will assume the vertices of G are given by x_1, \dots, x_M . Recall that $A = (a_{x_i x_j}) \equiv (a_{ij})$ is the transition matrix associated with G , where a_{ij} is the number of directed edges from vertex x_j to vertex x_i . Finally, write W_i for the set of all words in edges which are loops at vertex x_i , and put $P_i = P_{x_i}$, $Q_i = Q_{x_i}$.

Theorem 7.3. *Let G be a countable directed graph with tree top graph having a standard edge ordering. Let $A = (a_{ij})$ be the transition matrix for the graph, and let $k_i = \sum_{j=1}^M a_{ij}$ for $1 \leq i \leq M$. Then*

- (i) *Every eigenvector for \mathfrak{L}_G^* belongs to $P_i \mathcal{H}_G$ for some $1 \leq i \leq M$.*
- (ii) *The unit eigenvectors supported on $P_i \mathcal{H}_G$ are scalar multiples of the vectors*

$$\nu_{\lambda,i} = (1 - \|\lambda\|_2^2)^{1/2} \sum_{w \in W_i} \overline{w(\lambda)} \xi_w$$

where $\lambda = (\lambda_e)_{e \in E(G)} \in \mathbb{C}^n$, belongs to the set

$$\mathbb{B}_{G_{x_i}} \equiv (0_{k_1}, \dots, 0_{k_{i-1}}, \mathbb{D}_i, 0_{k_{i+1}}, \dots, 0_{k_M}) \in \mathbb{C}^n,$$

where $\mathbb{D}_i \equiv (\mathbb{B}_{a_{ii}}, 0_{k_i - a_{ii}}) \in \mathbb{C}^{k_i}$. Also,

$$\nu_{\lambda,i} = (1 - \|\lambda\|_2^2)^{1/2} \left(I - \sum_{e=x_i e x_i} \bar{\lambda}_e L_e \right)^{-1} \xi_{x_i}.$$

- (iii) *The eigenvectors $\nu_{\lambda,i}$ are supported on $Q_i P_i \mathcal{H}_G$ and are $Q_i \mathcal{H}_G$ -cyclic for \mathfrak{L}_G . They satisfy*

$$L_e^* \nu_{\lambda,i} = \bar{\lambda}_e \nu_{\lambda,i}$$

and if L_G is the n -tuple $L_G = (L_e)_{e \in E(G)}$, then $(p(L_G) \nu_{\lambda,i}, \nu_{\lambda,i}) = p(\lambda)$ for every polynomial $p = \sum_w a_w w$ in the semigroupoid algebra $\mathbb{C} \mathbb{F}^+(G)$. This extends to the map $\varphi_{\lambda,i}(A) = (A \nu_{\lambda,i}, \nu_{\lambda,i})$, which is a WOT-continuous multiplicative linear functional on \mathfrak{L}_G .

Proof. Let ν be an eigenvector for \mathfrak{L}_G^* . Then ν is an eigenvector for the projections P_1, \dots, P_M , but the only eigenvalues for a projection are 0 or 1. Thus, as the P_i have pairwise orthogonal ranges summing to the identity, there is a unique i with $\nu = P_i \nu \in P_i \mathcal{H}_G$.

There are scalars λ_e such that $L_e^* \nu = \bar{\lambda}_e \nu$ for all e . If $\nu = \sum_w \bar{a}_w \xi_w$, then

$$\sum_w \bar{\lambda}_e \bar{a}_w \xi_w = \bar{\lambda}_e \nu = L_e^* \nu = \sum_{w=ev} \bar{a}_{ev} \xi_v$$

and so $\bar{\lambda}_e \bar{a}_w = \bar{a}_{ew}$, and thus $\bar{a}_w = \overline{w(\lambda)}$. However, there will typically be λ_e equal to zero. If e is an edge with distinct initial and final

vertices, then $L_e^2 = 0$ because there are no words in $\mathbb{F}^+(G)$ of the form e^2w . Hence for such edges we have $\lambda_e = 0$. Further, let e be an edge in G with initial vertex distinct from x_i ; that is, $e = ex_j$ for $j \neq i$. Then $L_e = L_e P_j$, and

$$P_i(\overline{\lambda_e} \nu) = \overline{\lambda_e} \nu = L_e^* \nu = P_j L_e^* \nu = P_j(\overline{\lambda_e} \nu),$$

which can only happen if $\lambda_e = 0$. Therefore we have shown that the eigenvalues corresponding to eigenvectors supported on $P_i \mathcal{H}_G$ must belong to the set \mathbb{B}_{G_i} (As ν has finite norm, it follows that $\|\lambda\| < 1$), and the vectors are supported on $\text{span}\{\xi_w : w \in W_i\}$.

Now given $\lambda = (\lambda_e)_{e \in E(G)}$ in \mathbb{B}_{G_i} , we have

$$\left\| \sum_{e \in W_i} \overline{\lambda_e} L_e \right\|^2 = \left\| \sum_{e \in W_i} |\lambda_e|^2 L_e^* L_e \right\| \leq \sum_e |\lambda_e|^2 = \|\lambda\|^2 < 1,$$

so that $I - \sum_e \overline{\lambda_e} L_e$ is invertible, and its inverse is given by the power series

$$\left(I - \sum_e \overline{\lambda_e} L_e \right)^{-1} = \sum_{k \geq 0} \left(\sum_e \overline{\lambda_e} L_e \right)^k = \sum_{r(w)=x_i} \overline{w(\lambda)} L_w.$$

Thus the second identity for $\nu_{\lambda,i}$ follows, and from this it is clear that $\nu_{\lambda,i}$ is $Q_i \mathcal{H}_G$ -cyclic for \mathfrak{L}_G .

The vectors $\mu_{\lambda,i} = (1 - \|\lambda\|^2)^{-1/2} \nu_{\lambda,i}$ satisfy

$$L_e^* \mu_{\lambda,i} = L_e^* \sum_{r(w)=x_i} \overline{(ew)(\lambda)} \xi_{ew} = \overline{\lambda_e} \sum_{r(w)=x_i} \overline{w(\lambda)} \xi_w = \overline{\lambda_e} \mu_{\lambda,i},$$

and we also have $(L_w \nu_{\lambda,i}, \nu_{\lambda,i}) = w(\lambda) \|\nu_{\lambda,i}\|^2 = w(\lambda)$, which easily extends to polynomials by linearity. It is clear that $\varphi_{\lambda,i}$ is multiplicative and WOT-continuous. \blacksquare

Remark 7.4. There is an analogous version of this result for the eigenvectors of \mathfrak{R}_G^* , with the operators $\{R_e, Q_x\}$ in place of $\{L_e, P_x\}$. This fact is used in the proof of Theorem 9.4. Observe that the form of non-vacuum eigenvectors $\nu_{\lambda,x} = P_x Q_x \nu_{\lambda,x}$ for \mathfrak{R}_G^* are also determined by the a_{xx} loop edges over vertex x .

8. BEURLING THEOREM AND PARTIAL ISOMETRIES

In this section we establish a Beurling-type invariant subspace theorem for \mathfrak{L}_G . As a consequence we obtain a structure theorem for partial isometries in \mathfrak{L}_G , and an inner-outer factorization for elements of \mathfrak{L}_G .

We will say that a non-zero subspace \mathcal{W} of \mathcal{H} is *wandering* for \mathfrak{L}_G if the subspaces $L_w \mathcal{W}$ are pairwise orthogonal for distinct w in $\mathbb{F}^+(G)$.

Observe that the partial isometries L_w with $w \in \mathbb{F}^+(G)$ include the vertex projections $P_x = L_x$. Further, since the L_e are partial isometries we cannot ‘peel off’ L_e ’s when comparing the subspaces $L_w\mathcal{W}$, as is done in the case of isometries with orthogonal ranges. Nonetheless, equations such as $L_w^*L_w = P_{s(w)}$ give us a computational device for this comparison process.

Every wandering subspace \mathcal{W} generates an \mathfrak{L}_G -invariant subspace

$$\mathfrak{L}_G[\mathcal{W}] = \sum_{w \in \mathbb{F}^+(G)} \oplus L_w \mathcal{W}.$$

Every \mathfrak{L}_G -wandering vector ζ generates the cyclic invariant subspace $\mathfrak{L}_G[\zeta]$. We will say $\mathfrak{L}_G[\zeta]$ is a *minimal cyclic subspace* if $P_x\zeta = \zeta$ for some $x \in V(G)$. It is easy to see that given a wandering vector ζ , each vector $P_x\zeta$ which is non-zero is wandering as well.

Theorem 8.1. *Every invariant subspace of \mathfrak{L}_G is generated by a wandering subspace, and is the direct sum of minimal cyclic subspaces generated by wandering vectors. Every minimal cyclic invariant subspace generated by a wandering vector is the range of a partial isometry in \mathfrak{R}_G , and the choice of partial isometry is unique up to a scalar multiple.*

Proof. Let \mathcal{M} be a non-zero invariant subspace for \mathfrak{L}_G and form the subspace

$$\mathcal{W} = \mathcal{M} \ominus \left(\sum_{e \in E(G)} \oplus L_e \mathcal{M} \right).$$

First note that \mathcal{W} is a wandering subspace for \mathfrak{L}_G . To see this, let ξ and η be vectors in \mathcal{W} and let v, w belong to $\mathbb{F}^+(G)$. Consider the inner product $(L_w\xi, L_v\eta)$. This is clearly zero if v and w are distinct units in $V(G)$. If $w = x$ is a unit and $|v| \geq 1$, then $(P_x\xi, L_v\eta) = 0$ if $r(v) \neq x$, and otherwise $(\xi, L_v\eta) = 0$ by the definition of \mathcal{W} . If v and w are non-units with differing left most letters, then $(L_w\xi, L_v\eta) = 0$. Otherwise we would have $w = ew_1$ and $v = ev_1$ so that $(L_w\xi, L_v\eta) = ((L_e^*L_e)L_{w_1}\xi, L_{v_1}\eta)$. Since $L_e^*L_e = P_x$ for some x , we may repeat this argument to show this inner product is always zero.

We claim that $\mathcal{M} = \mathfrak{L}_G[\mathcal{W}]$. Let \mathcal{N} be the orthogonal complement of $\mathfrak{L}_G[\mathcal{W}]$ inside \mathcal{M} . Let $\eta \in \mathcal{N}$ and let $\xi = \sum_w L_w\zeta_w$, with each $\zeta_w \in \mathcal{W}$, belong to $\mathfrak{L}_G[\mathcal{W}]$. Then for all $e = ex$ in $E(G)$ we have

$$(L_e\eta, \xi) = \sum_w (L_e\eta, L_w\zeta_w) = \sum_{u \in \mathbb{F}^+(G)} (\eta, P_x L_u\zeta_w) = 0,$$

from the definition of \mathcal{N} . Further, $(P_x\eta, \xi) = (\eta, P_x\xi) = 0$ since $P_x\xi \in \mathfrak{L}_G[\mathcal{W}]$. Thus it follows that \mathcal{N} is invariant for \mathfrak{L}_G . Now let η belong

to the orthogonal complement of $\sum_e \oplus L_e \mathcal{N}$ inside \mathcal{N} . As η belongs to \mathcal{N} , we know that $(\eta, L_e \xi) = 0$ for all $\xi \in \mathfrak{L}_G[\mathcal{W}] \subseteq \mathcal{M}$. It follows that η belongs to \mathcal{W} . Indeed, let $\zeta \in \mathcal{M}$ and put $\zeta = \zeta_1 + \zeta_2$ with $\zeta_1 \in \mathcal{N}$ and $\zeta_2 \in \mathfrak{L}_G[\mathcal{W}]$. Then $L_e \zeta_1 \in \mathcal{N}$ and $L_e \zeta_2 \in \mathfrak{L}_G[\mathcal{W}]$ so that $(\eta, L_e \zeta) = (\eta, L_e \zeta_1) + (\eta, L_e \zeta_2) = 0$. Thus we have established that η is a vector in \mathcal{N} which is also in $\mathcal{W} \subseteq (\mathcal{N})^\perp$, whence $\eta = 0$. In particular, we have $\mathcal{N} = \sum_e \oplus L_e \mathcal{N}$. Finally, assume that $\mathcal{N} \neq \{0\}$ and let k_0 be minimal with $E_{k_0} \mathcal{N} \neq \{0\}$. Then

$$E_{k_0} \mathcal{N} \subseteq \sum_e E_{k_0} L_e \mathcal{N} = \sum_e L_e E_{k_0-1} \mathcal{N} = 0.$$

This contradiction yields $\mathcal{N} = \{0\}$, and hence $\mathcal{M} = \mathfrak{L}_G[\mathcal{W}]$ as claimed.

Next we observe that \mathcal{M} is the direct sum of cyclic subspaces. First note that $P_x \mathcal{W} \subseteq \mathcal{W}$ for all $x \in V(G)$ as noted in the discussion preceding the theorem. For each x , let $\{\zeta_{x,k_x}\}$ be an orthonormal basis for $P_x \mathcal{W} \subseteq \mathcal{W} = \sum_{x \in V(G)} \oplus P_x \mathcal{W}$. Then

$$\mathfrak{L}_G[\mathcal{W}] = \sum_x \sum_{k_x} \oplus \mathfrak{L}_G[\zeta_{x,k_x}].$$

The right-hand side of this identity is contained in $\mathcal{M} = \mathfrak{L}_G[\mathcal{W}]$ by \mathfrak{L}_G -invariance. On the other hand, note that for distinct wandering vectors ζ_{x,k_x} and ζ_{y,l_y} in \mathcal{W} (with x and y not necessarily distinct), the cyclic subspaces $\mathfrak{L}_G[\zeta_{x,k_x}]$ and $\mathfrak{L}_G[\zeta_{y,l_y}]$ are perpendicular. Lastly, it is clear that vectors in $\mathfrak{L}_G[\mathcal{W}]$ belong to the sum on the right side.

The subspace \mathcal{M} is minimal cyclic if and only if \mathcal{W} is one-dimensional and there is an x with $P_x \mathcal{W} = \mathcal{W}$ and $P_y \mathcal{W} = 0$ for $y \neq x$. Consider $\mathcal{M} = \mathfrak{L}_G[\zeta]$, where $P_x \zeta = \zeta$ is a unit \mathfrak{L}_G -wandering vector. Define a linear transformation R_ζ on \mathcal{H}_G by the rule $R_\zeta \xi_w = L_w \zeta$ for $w \in \mathbb{F}^+(G)$. Then for $w \neq v$ in $\mathbb{F}^+(G)$ we have $(R_\zeta \xi_w, R_\zeta \xi_v) = (L_w \zeta, L_v \zeta) = 0$ because ζ is wandering. Further, when $L_w^* L_w \neq P_x$ a similar computation shows that $\|R_\zeta \xi_w\|^2 = 0$. Moreover, if $L_w^* L_w = P_x$ we see that $\|R_\zeta \xi_w\|^2 = \|P_x \zeta\|^2 = 1$. Thus it follows that the operator R_ζ is a partial isometry with range equal to \mathcal{M} by design. Finally, observe that for each edge e and w in $\mathbb{F}^+(G)$

$$R_\zeta L_e \xi_w = R_\zeta \xi_{ew} = L_{ew} \zeta = L_e R_\zeta \xi_w.$$

Similarly, $R_\zeta P_y \xi_w = L_{yw} \zeta = P_y R_\zeta \xi_w$ and we have $R_\zeta \in \mathfrak{L}'_G = \mathfrak{R}_G$ as required.

To verify the uniqueness assertion, suppose $P_x \zeta = \zeta$ is a unit \mathfrak{L}_G -wandering vector and $\mathcal{M} = \mathfrak{L}_G[\zeta] = R_\zeta \mathcal{H}_G = R \mathcal{H}_G$ is the range of another partial isometry R in \mathfrak{R}_G . We claim that $R = \lambda R_\zeta$ for some $|\lambda| = 1$. First observe that the vectors $R \xi_w = L_w R \xi_x$, where $s(w) = x$,

form an orthonormal basis for $\mathcal{M} = R\mathcal{H}_G$. This follows from Corollary 8.6 since the initial projection $R^*R = Q_x$. (Note that Corollary 8.6 relies on Theorem 8.5, which in turn uses part of the proof of this theorem, but that there is no circular logic. The proof of Theorem 8.5 does not use the uniqueness from this theorem.) In particular, clearly $R\xi_x$ belongs to the wandering subspace $\mathcal{W} = P_x\mathcal{W} = \text{span}\{\zeta\}$, and hence $R\xi_x = \lambda\zeta = \lambda R_\zeta\xi_x$ for some $|\lambda| = 1$. Thus for $w = wx$ we have $R\xi_w = L_w R\xi_x = \lambda R_\zeta\xi_w$, and it follows that $R = RQ_x = \lambda R_\zeta Q_x = \lambda R_\zeta$. \blacksquare

Remark 8.2. There is obviously an analogue of this result for the invariant subspaces of \mathfrak{R}_G , where the notion of wandering is determined by the R_e and Q_x . This is used in Theorem 9.4 and Section 10. We also note that Theorem 8.1 parallels the Beurling Theorem from [29], and gives a slight improvement for these algebras. Indeed, we have identified the minimal cyclic subspaces as ranges of partial isometries in the commutant algebra \mathfrak{R}_G , and shown that the decomposition in terms of minimal invariant subspaces is unique. Further, in Theorem 8.5 we prove that all such operators have a standard form.

The range of every partial isometry R in \mathfrak{R}_G is cyclic since $R\mathcal{H}_G = \overline{R\mathfrak{L}_G\xi_\phi} = \overline{\mathfrak{L}_G R\xi_\phi}$, where $\xi_\phi = \sum_{x_k \in V(G)} \frac{1}{k}\xi_{x_k}$. However, we observe through the next example that these subspaces are not necessarily minimal cyclic. This is different from the free semigroup case [9, 31], where ranges of isometries are minimal cyclic subspaces. The basic difference here is that partial isometries in the commutant can ‘cross-over’ between distinct tree components.

Example 8.3. Let G be the directed graph with transition matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Let $V(G) = \{x_1, x_2\}$ and let $e_1 = x_1e_1x_1$, $e_2 = x_2e_2x_1$ and $e_3 = x_1e_3x_2$. Let $R \in \mathfrak{R}_G$ be the isometry defined by $R = R_{e_1} + R_{e_2}$. It is an isometry because $R_{e_1}Q_2 = 0 = R_{e_2}Q_1$, whereas the ranges of $R_{e_1}Q_1 = R_{e_1}$ and $R_{e_2}Q_2 = R_{e_2}$ are orthogonal. The range of R is a cyclic subspace given by

$$\mathcal{M} = R\mathcal{H}_G = \text{span}\{\xi_{we_1} + \xi_{we_2} : w \in \mathbb{F}^+(G)\} = \mathfrak{L}_G[\xi_{e_1} + \xi_{e_2}].$$

Thus, $L_e\mathcal{M} = \text{span}\{\xi_{ewe_1} + \xi_{ewe_2} : w \in \mathbb{F}^+(G)\}$ for $e \in E(G)$, and hence

$$\mathcal{W} = \mathcal{M} \ominus \left(\sum_e \oplus L_e\mathcal{M} \right) = \text{span}\{\xi_{e_1}, \xi_{e_2}\}.$$

Thus \mathcal{W} is two-dimensional here with $P_i\mathcal{W}$ spanned by ξ_{e_i} for $i = 1, 2$. In particular, from the proof of Theorem 8.1, we see that \mathcal{M} is the direct sum of two minimal cyclic subspaces $\mathcal{M} = \mathfrak{L}_G[\xi_{e_1}] \oplus \mathfrak{L}_G[\xi_{e_2}]$.

We next derive an explicit characterization of partial isometries in \mathfrak{L}_G . We begin with a computational lemma.

Lemma 8.4. *For $A \in \mathfrak{L}_G$, we have*

$$(2) \quad R_f^* A - A R_f^* = (R_f^* A \xi_x) \otimes \xi_x^* \quad \text{where} \quad s(f) = x.$$

Proof. Let $A \in \mathfrak{L}_G$. Then

$$A = AI = A \left(\sum_{x \in V(G)} Q_x \right) = A \left(\sum_x (\xi_x \otimes \xi_x^* + \sum_{s(e)=x} R_e R_e^*) \right).$$

Hence for $f \in E(G)$,

$$\begin{aligned} R_f^* A &= \sum_x (R_f^* A \xi_x) \otimes \xi_x^* + \sum_x \sum_{s(e)=x} R_f^* A R_e R_e^*, \\ &= \sum_x (R_f^* A \xi_x) \otimes \xi_x^* + A R_f^*, \end{aligned}$$

since $R_f^* A R_e R_e^* = R_f^* R_e A R_e^* = \delta_{f,e} A R_f^*$. Further, $A \xi_x = Q_x A \xi_x$ so that $R_f^* A \xi_x = 0$ when $s(f) \neq x$. The identity (2) follows. \blacksquare

Theorem 8.5. *Let V be a partial isometry in \mathfrak{L}_G . Then*

$$V = \sum_{i \in \mathcal{I}} \oplus L_{\eta_i}$$

where $\{\eta_i\}_{i \in \mathcal{I}}$ are unit wandering vectors for \mathfrak{R}_G supported on distinct $Q_x \mathcal{H}_G$, and where the series converges in the strong operator topology if \mathcal{I} is an infinite set.

Proof. Let η_1, η_2, \dots be an orthonormal basis for the \mathfrak{R}_G -wandering subspace \mathcal{W} of $\mathcal{M} = V \mathcal{H}_G$, where each η_i belongs to some $Q_x \mathcal{H}_G$, as in the proof of the \mathfrak{R}_G version of Theorem 8.1. (In fact, it follows that the set $\{\eta_i\}_{i \in \mathcal{I}}$ is equal to the set of non-zero vectors amongst $\{V \xi_x : x \in V(G)\}$.) We show that $B_i = L_{\eta_i}^* V$ belongs to \mathfrak{L}_G by showing $R_f B_i - B_i R_f = 0$ for all $f \in E(G)$ and $R_x B_i - B_i R_x = 0$ for all vertices $x \in V(G)$. From the previous lemma, if $s(f) = x$, then

$$R_f^* L_{\eta_i} = (R_f^* L_{\eta_i} \xi_x) \otimes \xi_x^* + L_{\eta_i} R_f^*,$$

and so

$$R_f L_{\eta_i}^* = -\xi_x \otimes (R_f^* L_{\eta_i} \xi_x)^* + L_{\eta_i}^* R_f.$$

Consequently,

$$\begin{aligned} R_f B_i - B_i R_f &= R_f L_{\eta_i}^* V - L_{\eta_i}^* V R_f = (R_f L_{\eta_i}^* - L_{\eta_i}^* R_f) V \\ &= \left(-\xi_x \otimes (R_f^* L_{\eta_i} \xi_x)^* \right) V = -\xi_x \otimes (V^* R_f^* L_{\eta_i} \xi_x)^*. \end{aligned}$$

As each η_i belongs to $\mathcal{M}_x = Q_x \mathcal{M}$ for some x , we have $L_{\eta_i} \xi_x = 0$ if η_i is not in \mathcal{M}_x . On the other hand, if $\eta_i \in \mathcal{M}_x$, then $L_{\eta_i} \xi_x = \eta_i$ and this vector is orthogonal to $R_f V \mathcal{H}_G$ for each edge f . Hence $V^* R_f^* L_{\eta_i} \xi_x = 0$ in this case. So we obtain $R_f B_i - B_i R_f = 0$ for all f . Similarly, $R_x B_i - B_i R_x = 0$ for all x since $B_i = L_{\eta_i}^* V$ is a product of operators which commute with the projections $R_x = Q_x$. It follows that $B_i \in \mathfrak{L}_G$.

We now have $L_{\eta_i}^* V = B_i$ with $B_i \in \mathfrak{L}_G$. Also, since the range of L_{η_i} is contained in the range of V , it follows that B_i is a partial isometry with range equal to the range of $L_{\eta_i}^*$. But this range is the initial space of L_{η_i} , which is easily seen to be P_x when $\eta_i \in \mathcal{M}_x$. Thus $P_x B_i = B_i \in \mathfrak{L}_G$ is a partial isometry with $\xi_x \in \text{Ran}(B_i)$. We claim that $B_i = \lambda P_x$ for some $|\lambda| = 1$. Indeed, there is a vector $\xi = B_i^* B_i \xi$, $\|\xi\| = 1$, such that $B_i \xi = \xi_x$ and by considering the Fourier expansion for $B_i \in \mathfrak{L}_G$ we can see that $\xi = B_i^* \xi_x = \bar{\lambda} \xi_x$ where $\lambda = (B_i \xi_x, \xi_x)$.

Therefore $B_i \xi_x = \lambda \xi_x$ so that $B_i \xi_w = R_w B_i \xi_x = \lambda \xi_w$ for $w = xw$ and hence $B_i P_x = \lambda P_x$. But $B_i P_y = P_x B_i P_y = 0$ for $y \neq x$, because B_i is a partial isometry and otherwise we would have $\|B_i^* P_x\| > 1$ (a contradiction since $B_i^* P_x = \bar{\lambda} P_x$). The claim now follows because

$$B_i = B_i I = B_i \left(\sum_{y \in V(G)} P_y \right) = B_i P_x = \lambda P_x.$$

Evidently $(L_{\eta_i} L_{\eta_i}^*) V = \lambda L_{\eta_i} = L_{\eta'_i}$ where $\eta'_i = \lambda \eta_i$. As the projections $L_{\eta_i} L_{\eta_i}^*$ are orthogonal and sum to $V V^*$ it follows that

$$V = V V^* V = \sum_i L_{\eta_i} L_{\eta_i}^* V = \sum_i L_{\eta'_i}.$$

Finally, each vector η'_i is supported on some $Q_x \mathcal{H}_G$ from the Beurling Theorem. But $L_{\eta'_i} \xi_x = Q_x \eta'_i$ by definition. Thus, as $V = \sum_i L_{\eta'_i}$ is a partial isometry it follows that the vectors $\{\eta'_i\}$ are supported on distinct $Q_x \mathcal{H}_G$. \blacksquare

As an immediate consequence we obtain the following simple description of initial projections. This result will be useful in Section 10.

Corollary 8.6. *If V is a partial isometry in \mathfrak{L}_G , then the initial projection of V is given by*

$$V^* V = \sum_{x \in \mathcal{I}} P_x \quad \text{where} \quad \mathcal{I} = \{x \in V(G) : V \xi_x \neq 0\}.$$

Lastly, we obtain an inner-outer factorization for elements of \mathfrak{L}_G which generalizes the H^∞ [13, 16] and \mathfrak{L}_n cases [9, 31]. Given a subset $\mathcal{S} \subseteq V(G)$ of vertices, define the \mathcal{S} -inner elements of \mathfrak{L}_G to be the partial isometries with initial projection $\sum_{x \in \mathcal{S}} P_x$. Also define the

\mathcal{S} -outer elements to be those elements of \mathfrak{L}_G with range dense inside $\sum_{x \in \mathcal{S}} P_x \mathcal{H}_G$.

Corollary 8.7. *Every A in \mathfrak{L}_G factors as $A = VB$ where V is an \mathcal{S} -inner element of \mathfrak{L}_G and B is \mathcal{S} -outer inside \mathfrak{L}_G with $\mathcal{S} = \{x \in V(G) : A\xi_x \neq 0\}$.*

Proof. Let $\mathcal{M} = \overline{\text{Ran}(A)} = \overline{A\mathfrak{R}_G\xi_\phi} = \overline{\mathfrak{R}_G A\xi_\phi}$ and let $\mathcal{S} = \{x \in V(G) : A\xi_x \neq 0\}$. Then there are unit \mathfrak{R}_G -wandering vectors $\eta_x = Q_x \eta_x$ for $x \in \mathcal{S}$ such that $\mathcal{M} = \sum_{x \in \mathcal{S}} \oplus L_{\eta_x} \mathcal{H}_G$. Let $V = \sum_{x \in \mathcal{S}} \oplus L_{\eta_x}$ and observe that $V^*V = \sum_{x \in \mathcal{S}} L_{\eta_x}^* L_{\eta_x} = \sum_{x \in \mathcal{S}} P_x$. Let $B_x = L_{\eta_x}^* A$ for $x \in \mathcal{S}$ and put $B = V^*A = \sum_{x \in \mathcal{S}} B_x$. It is clear that $A = VB$, and that each B_x has dense range in $P_x \mathcal{H}_G$. Further, since the L_{η_x} have pairwise orthogonal ranges which span \mathcal{M} , it follows that B has range dense inside $\sum_{x \in \mathcal{S}} P_x \mathcal{H}_G$. To complete the proof it suffices to show that each B_x belongs to $\mathfrak{R}_G = \mathfrak{L}_G$, and for this we may employ Lemma 8.4 as in the proof of Theorem 8.5.

We mention there is also a uniqueness associated with the factorization $A = VB = \sum_{x \in \mathcal{S}} L_{\eta_x} B_x$. The factors L_{η_x} and B_x of $A_x = L_{\eta_x} B_x$ are unique up to a scalar multiple since the wandering vectors $\eta \in Q_x(\mathcal{M} \ominus \sum_e \oplus R_e \mathcal{M})$, $x \in \mathcal{S}$, are unique up to a scalar. ■

9. CLASSIFICATION AND AUTOMORPHISMS

In this section we establish a classification theorem for the algebras \mathfrak{L}_G by showing that G is a complete unitary invariant for \mathfrak{L}_G . Our analysis also yields a large class of unitarily implemented automorphisms of the algebras which act transitively on sets of eigenvectors of \mathfrak{L}_G^* .

Theorem 9.1. *Let G and G' be countable directed graphs. Then the following assertions are equivalent.*

- (i) G and G' are isomorphic.
- (ii) \mathfrak{L}_G and $\mathfrak{L}_{G'}$ are unitarily equivalent.

The proof relies on properties of \mathfrak{L}_G which are interesting in their own right. We begin by showing that the family of vertex projections is a unitary invariant. For $k \geq 0$, recall that E_k is the projection onto $\text{span}\{\xi_w : |w| = k\}$. Hence E_0 is the projection onto the vacuum space $\text{span}\{\xi_x : x \in V(G)\}$, and it is clear that

$$(3) \quad E_0 = \sum_{x \in V(G)} \xi_x \otimes \xi_x^* = I - \sum_{e \in E(G)} L_e L_e^*.$$

Hence the rank one projections $\xi_x \otimes \xi_x^*$ are obtained by compressing to $Q_x \mathcal{H}_G$

$$(4) \quad \xi_x \otimes \xi_x^* = Q_x - \sum_{e \in E(G)} L_e L_e^* Q_x,$$

since the projections $\{Q_x : x \in V(G)\}$ are reducing for \mathfrak{L}_G . The families $\{R_e : e \in E(G)\}$ and $\{P_x : x \in V(G)\}$ have a similar relationship.

Lemma 9.2. *The projections $\{Q_x : x \in V(G)\}$ form the unique maximal family of non-zero pairwise orthogonal irreducible projections for \mathfrak{L}_G . The projections $\{P_x : x \in V(G)\}$ play the same role for \mathfrak{R}_G .*

Proof. As $I = \sum_x \oplus Q_x$, the projections $\{Q_x\}$ form a maximal family of pairwise orthogonal reducing projections for \mathfrak{L}_G . To see minimality suppose $0 \neq Q \leq Q_x$ is an \mathfrak{L}_G -reducing projection. From equation (4) above, Q commutes with $\xi_x \otimes \xi_x^*$ so the vector ξ_x either belongs to $Q\mathcal{H}_G$ or is orthogonal to it. However, by the \mathfrak{L}_G^* -invariance of $Q \neq 0$ there is clearly a $\xi = Q\xi$ such that $(\xi, \xi_x) \neq 0$. Hence $\xi_x \in Q\mathcal{H}_G$, and \mathfrak{L}_G -invariance gives $Q = Q_x$. Thus each Q_x is an irreducible projection.

To observe uniqueness of the family, suppose projections $\{Q_j\}$ form another maximal such family. The Q_j are non-zero projections in $\mathfrak{L}'_G = \mathfrak{R}_G$, hence by Corollary 4.6 each Q_j belongs to the linear span of the family $\{Q_x\}$. Thus Q_j is equal to the sum of a subset of these projections, and by the irreducibility of Q_j , it follows that in fact $Q_j = Q_x$ for some x . Maximality of the Q_j family ensures every Q_x is obtained in this manner. It follows that the family of projections Q_j must actually be the family of Q_x . The same proof works for \mathfrak{R}_G and the family $\{P_x\}$. \blacksquare

Our next step is to show how the number of directed edges between pairs of vertices in G can be computed in terms of \mathfrak{L}_G and the vacuum vectors. Let \mathfrak{L}_G^0 be the WOT-closed two-sided ideal of \mathfrak{L}_G generated by the L_e ;

$$\mathfrak{L}_G^0 = \langle L_e : e \in E(G) \rangle.$$

Consideration of Fourier expansions in \mathfrak{L}_G shows that every $A \in \mathfrak{L}_G^0$ satisfies $(A\xi_x, \xi_x) = 0$ for each x , and in fact this condition characterizes \mathfrak{L}_G^0 . This ideal helps to identify the directed graph structure of G from the algebra \mathfrak{L}_G .

Lemma 9.3. *Let a_{yx} be the number of directed edges $e = yex$ in G from vertex x to vertex y . Then*

$$(5) \quad a_{yx} = \text{rank}(P_y E_1 Q_x) = \dim \left[P_y (\mathfrak{L}_G^0 \xi_x \ominus (\mathfrak{L}_G^0)^2 \xi_x) \right].$$

Furthermore, the ideal \mathfrak{L}_G^0 may be computed as the set

$$\mathfrak{L}_G^0 = \{A \in \mathfrak{L}_G : A^* \xi_x = 0 \text{ for } x \in V(G)\}.$$

Proof. The family $\{P_y, E_1, Q_x\}$ is mutually commuting, hence the projection $P_y E_1 Q_x$ acts on basis vectors by $P_y E_1 Q_x \xi_w = E_1 \xi_{ywx}$. Thus it is the range projection for the subspace

$$P_y E_1 Q_x \mathcal{H}_G = \text{span}\{\xi_e : e = yex\}.$$

This yields the identity $a_{yx} = \text{rank}(P_y E_1 Q_x)$. From the preceding discussion, every $A \in \mathfrak{L}_G^0$ has an expansion of the form $A \sim \sum_{|w| \geq 1} a_w L_w$, and acts on basis vectors by $A \xi_v = \sum_{|w| \geq 1} a_w \xi_{vw}$. It follows that the subspace

$$\mathfrak{L}_G^0 \xi_x = \text{span}\{\xi_w : s(w) = x \text{ and } |w| \geq 1\},$$

whereas

$$(\mathfrak{L}_G^0)^2 \xi_x = \text{span}\{\xi_w : s(w) = x \text{ and } |w| \geq 2\}.$$

Therefore we have the subspace equalities

$$\begin{aligned} P_y (\mathfrak{L}_G^0 \xi_x \ominus (\mathfrak{L}_G^0)^2 \xi_x) &= \text{span}\{\xi_w : w = ywx \text{ and } |w| = 1\} \\ &= P_y E_1 Q_x \mathcal{H}_G. \end{aligned}$$

To see the alternate description of \mathfrak{L}_G^0 in terms of vacuum vectors, notice that if $A \sim \sum_{|w| \geq 1} a_w L_w$ belongs to \mathfrak{L}_G^0 , then clearly $A^* \xi_x = 0$ for $x \in V(G)$. On the other hand, if $A \sim \sum_w a_w L_w$ belongs to \mathfrak{L}_G and A^* annihilates each vacuum vector, then

$$0 = (\xi_x, A^* \xi_x) = (A \xi_x, \xi_x) = a_x \text{ for } x \in V(G),$$

and A belongs to \mathfrak{L}_G^0 . ■

We now prove the classification theorem. The proof makes use of Theorem 9.4, which we present below because of its independent interest. But notice that this theorem is not needed for the special case of graphs with no loop edges at vertices; in this case Proposition 7.2 shows that the ideal \mathfrak{L}_G^0 is invariant under unitary isomorphism.

Proof of Theorem 9.1. If G and G' are isomorphic, then there is a relabelling map between the vertices and edges of the graphs which clearly induces a unitary operator between \mathcal{H}_G and $\mathcal{H}_{G'}$. This unitary intertwines the generators $\{L_e, P_x\}$ of \mathfrak{L}_G with the generators $\{L_{e'}, P'_x\}$ of $\mathfrak{L}_{G'}$, and *a fortiori* the algebras \mathfrak{L}_G and $\mathfrak{L}_{G'}$ are unitarily equivalent.

Conversely, suppose there is a unitary operator $U : \mathcal{H}_{G'} \rightarrow \mathcal{H}_G$ for which $U^* \mathfrak{L}_G U = \mathfrak{L}_{G'}$. Without loss of generality, we may assume that $\mathcal{H}_G = \mathcal{H}_{G'}$. By Lemma 9.2 the family $\{P_x\}$ is a unitary invariant of \mathfrak{L}_G , hence the number of vertices in G and G' is the same and we

may assume, perhaps after reordering, that $U^*P_xU = P'_x$ for each x in $V(G) = V(G')$. Further, under this unitary equivalence the vacuum vectors ξ'_x for $\mathfrak{L}_{G'}$ are mapped to eigenvectors for \mathfrak{L}_G^* . But Theorem 9.4 shows that these vectors may be moved to the vacuum vectors ξ_x by a unitary which implements an automorphism of \mathfrak{L}_G . In particular, we may also assume that $U\xi'_x = \xi_x$ for each x . It remains to show that the number a_{yx} of directed edges $e = yex$ in G is equal to the number a'_{yx} of directed edges $e' = ye'x$ in G' , for all $x, y \in V(G) = V(G')$. This readily follows from Lemma 9.3 and our above assumptions; indeed, we have $U^*\mathfrak{L}_G^0U = \mathfrak{L}_{G'}^0$ and

$$\begin{aligned} a'_{yx} &= \dim \left[P'_y(\mathfrak{L}_{G'}^0\xi'_x \ominus (\mathfrak{L}_{G'}^0)^2\xi'_x) \right] \\ &= \dim \left[U^*P_y(\mathfrak{L}_G^0\xi_x \ominus (\mathfrak{L}_G^0)^2\xi_x) \right] = a_{yx}. \end{aligned}$$

Therefore the directed graphs G and G' are isomorphic. \blacksquare

The proof of the classification theorem relies on the existence of certain unitarily implemented automorphisms of \mathfrak{L}_G which we now discuss. Let $\nu = \nu_{\lambda,x} = P_xQ_x\nu_{\lambda,x}$ be an eigenvector for the algebra \mathfrak{R}_G^* , as given in the \mathfrak{R}_G version of Theorem 7.3 (see Remark 7.4). We show there is an automorphism of \mathfrak{L}_G which is implemented by a unitary which maps $\nu_{\lambda,x}$ to $\nu_{0,x} = \xi_x$. This transitive action of unitary automorphisms was obtained for free semigroup algebras by Davidson and Pitts [10] and our proof is an elaboration and generalization of their analysis.

Recall first that in the case of the free semigroup algebra \mathfrak{L}_n and its commutant algebra \mathfrak{R}_n , with eigenvector η for \mathfrak{R}_n^* , the \mathfrak{R}_n -wandering subspace \mathcal{M} for the \mathfrak{R}_n -invariant subspace $\{\eta\}^\perp$ has an orthonormal basis consisting of n vectors $\{\eta_1, \dots, \eta_n\}$ say. Also recall that if a vector η is \mathfrak{R}_n -wandering then one can define the isometry L_η in \mathfrak{L}_G by the specification $L_\eta\xi_w = R_w\eta$ for all words w . The desired automorphism in this case is in fact effected by the correspondence $L_{e_i} \rightarrow L_{\eta_i}$, for $1 \leq i \leq n$, where e_1, \dots, e_n are the n loop edges of the free semigroup graph. Using only the fact that the wandering space for \mathcal{M} is n -dimensional, we shall develop a similar argument from first principles to define the desired automorphism of \mathfrak{L}_G . There are some complications in the new setting. In particular, to capture all the generators we must consider more than the wandering subspace for $\{\nu\}^\perp$.

For the next theorem we use the following notation. The set W_x is the collection of words in the loop edges at vertex x , and \mathcal{H}_x is the closed span of the basis vectors ξ_w for w in W_x . Also we identify \mathcal{H}_x with the Fock space for \mathfrak{L}_n and identify the generators L_1, \dots, L_n (respectively R_1, \dots, R_n) of \mathfrak{L}_n (respectively \mathfrak{R}_n) with the restrictions

$L_e|_{\mathcal{H}_x}$ (respectively $R_e|_{\mathcal{H}_x}$) for the n edges e with $xex = e$. In particular, from our earlier notation we have $n = a_{xx}$.

Theorem 9.4. *Let G be a countable directed graph, and let $\nu = \nu_{\lambda,x} = Q_x P_x \nu_{\lambda,x}$ be an eigenvector for \mathfrak{R}_G^* .*

(i) *The subspace*

$$\mathcal{M} = \{\nu\}^\perp \bigcap \left(\bigcap_{y \neq x} \{\xi_y\}^\perp \right)$$

is \mathfrak{R}_G -invariant with wandering subspace basis $\{\eta_e : e \in E(G)\}$ where $\eta_e = \xi_e$ if $r(e) \neq x$, where $\eta_e = R_e \nu$ if $r(e) = x$ and $s(e) \neq x$, and where $\{\eta_e : xex = e\}$ is a basis for the \mathfrak{R}_n -wandering subspace of

$$\{\nu\}^\perp \cap \mathcal{H}_x \quad \text{for} \quad \{R_j\} = \{R_e|_{\mathcal{H}_x} : xex = e\}.$$

- (ii) *The correspondence $e \rightarrow L_{\eta_e}$, and $x \rightarrow L_x$ gives a purely atomic free partial isometry representation of G satisfying the multiplicity one condition at each vertex, and the correspondence $L_e \rightarrow L_{\eta_e}$ extends to an automorphism of \mathfrak{L}_G .*
- (iii) *Let $S_e = L_{\eta_e}$, for $e \in E(G)$, and $S_y = L_y$ for $y \in V(G)$. Then there is a unitary operator W defined by*

$$W\xi_w = \begin{cases} w(S)\nu & \text{if } s(w) = x \\ w(S)\xi_y & \text{if } s(w) = y, y \neq x. \end{cases}$$

Furthermore, $\text{Ad } W(L_e) = WL_e W^ = S_e$ for $e \in E(G)$ and $\text{Ad } W$ is the automorphism of \mathfrak{L}_G given in (ii).*

Proof. Choose $\eta_{e_1}, \dots, \eta_{e_m}$ to be an orthonormal basis (possibly countably infinite) for the \mathfrak{R}_n -wandering subspace of $\{\nu\}^\perp \cap \mathcal{H}_x$, and let η_e , for the other edges of G , be specified as in (i). We claim this basis spans the \mathfrak{R}_G -wandering subspace for \mathcal{M} which is $\mathcal{M} \ominus (\sum_e \oplus R_e \mathcal{M})$. Indeed, from the definition of \mathcal{M} it is easy to see that each of the three types of basis vector belong to this wandering subspace. On the other hand, by the choice of η_e for e with $e = xex$ it is not hard to see that all the non-zero vectors of the form $R_w \eta_e$ give rise to an orthonormal basis for \mathcal{M} . Thus the wandering subspace for \mathcal{M} has orthonormal basis $\{\eta_e : e \in E(G)\}$. Further, each η_e is supported on a particular Q_x , hence the decomposition $\mathcal{M} = \sum_{e \in E(G)} \oplus \mathfrak{R}_G[\eta_e]$ really is the decomposition of \mathcal{M} into minimal cyclic subspaces indicated in the \mathfrak{R}_G version of Theorem 8.1 (see Remark 8.2).

We establish (ii) and (iii) together. The operators $S_e = L_{\eta_e}$ have the defining property $S_e \xi_w = R_w \eta_e$, for all w in $\mathbb{F}^+(G)$. It follows that S_e is a partial isometry with initial projection $P_y = L_y$ where $s(e) = y$ since

$\eta_e = Q_y \eta_e$. Moreover, the final projection of S_e is the space $\mathfrak{R}_G[\eta_e]$ by design and so, by our construction of the basis it is clear that

$$\begin{aligned} \sum_{r(e)=y} S_e S_e^* &= P_y - \xi_y \otimes \xi_y^*, \quad \text{for } y \neq x, \\ \sum_{r(e)=x} S_e S_e^* &= P_x - \nu \otimes \nu^*. \end{aligned}$$

It now follows that the map $e \rightarrow S_e$, $y \rightarrow S_y = L_y$ gives a free partial isometry representation of G satisfying the multiplicity one condition considered in Proposition 3.4 and that the generators $\{S_e, L_y\}$ are mutually unitarily equivalent to the generators $\{L_e, L_y\}$ by the unitary W .

It remains to show that the unitary automorphism $\text{Ad } W(X) = WXW^*$ satisfies $\text{Ad } W(\mathfrak{L}_G) = \mathfrak{L}_G$, that is, that the unitary automorphism $\text{Ad } W$ of $\mathcal{B}(\mathcal{H})$ restricts to an automorphism of \mathfrak{L}_G rather than an endomorphism. Using the gauge automorphisms of Section 6, at this point in the proof we can easily follow the free semigroup approach (see [10], Remark 4.13). Indeed, the algebra $\mathfrak{A} = \text{Ad } W(\mathfrak{L}_G)$ is contained in \mathfrak{L}_G , and hence $\nu_{0,x} = \xi_x$ is an eigenvector for \mathfrak{A}^* . Since \mathfrak{A} is unitarily equivalent to \mathfrak{L}_G , there is a non-zero μ such that $W\nu_{\mu,x} = \xi_x$. Hence we can apply the above argument again to obtain another unitary W' for which $\text{Ad } W'W(L_e) = S'_e$, where the S'_e are determined as above by an orthonormal basis \mathcal{B} for the wandering subspace of the subspace $\cap_x \{\xi_x\}^\perp$. Let U be a unitary in \mathcal{U}_m which intertwines the orthonormal set $\{\xi_e : e \in E(G)\}$ with the vectors of \mathcal{B} , in such a way that $UP_y = P_y$ for each y . Then it follows that $\text{Ad } W'W = \theta_U$, the gauge automorphism of \mathfrak{L}_G determined by U . Consequently, the two endomorphisms of \mathfrak{L}_G must actually be automorphisms. \blacksquare

We can immediately deduce the corresponding classification of the norm-closed algebras generated by the generators of \mathfrak{L}_G . Let us denote this algebra, which is a non-commutative version of the disc algebra, as \mathcal{A}_G . In the case of a finite directed graph this algebra was studied in the general framework of tensor algebras over correspondences by Muhly and Solel [28, 29], but the basic classification question was not considered. Recall that \mathcal{A}_G and $\mathcal{A}_{G'}$ are star-extendibly isomorphic if there is an isomorphism $\mathcal{A}_G \rightarrow \mathcal{A}_{G'}$ which is the restriction of a (necessarily unique) C^* -algebra isomorphism $C^*(\mathcal{A}_G) \rightarrow C^*(\mathcal{A}_{G'})$.

Corollary 9.5. *Let G, G' be countable directed graphs. Then the following assertions are equivalent.*

- (i) G and G' are isomorphic.

(ii) \mathcal{A}_G and $\mathcal{A}_{G'}$ are unitarily equivalent.

Also, if each vertex of G , G' has finite degree then a star-extendible isomorphism between \mathcal{A}_G and $\mathcal{A}_{G'}$ is unitarily implemented.

Proof. If \mathcal{A}_G and $\mathcal{A}_{G'}$ are unitarily equivalent, then so are their WOT-closures and so the equivalence of (i) and (ii) follows from Theorem 9.1. For the final assertion note that $C^*(\mathcal{A}_G)$ contains the collection of compact operators $K \in \mathcal{B}(\mathcal{H})$ such that $KQ_x = Q_xK$ for all vertices x . Indeed, for $x \in V(G)$ and words $v = vx$, $w = wx$ in $\mathbb{F}^+(G)$, equation (3) shows that the rank one projection

$$\begin{aligned} \xi_v \otimes \xi_w^* &= L_v (\xi_x \otimes \xi_x^*) L_w \\ &= L_v P_x \left(I - \sum_{e \in E(G)} L_e L_e^* \right) P_x L_w \\ &= L_v P_x L_w - \sum_{e \in E(G)} L_v P_x L_e L_e^* P_x L_w \end{aligned}$$

belongs to $C^*(\mathcal{A}_G)$. Note that the summation here is finite. Thus it follows that the isomorphism $C^*(\mathcal{A}_G) \rightarrow C^*(\mathcal{A}_{G'})$ is unitarily implemented, and hence the restriction to \mathcal{A}_G produces a unitary equivalence with $\mathcal{A}_{G'}$. \blacksquare

We finish this section with an example which may clarify some of the subtleties of Theorem 9.4.

Example 9.6. Let G be the directed graph with vertex set $V(G) = \{x_1, x_2\}$ and edges $e_i = x_1 e_i x_1$ for $i = 1, 2$, $e_3 = x_2 e_3 x_1$, and $e_4 = x_1 e_4 x_2$. Let $\nu = \nu_{\lambda,1} = P_1 Q_1 \nu_{\lambda,1}$ be an eigenvector for \mathfrak{R}_G^* . In this case $\mathcal{M} = \{\nu\}^\perp \cap \{\xi_2\}^\perp$. The orthonormal basis for the wandering space of \mathcal{M} from the theorem is given by $\eta_{e_3} = \xi_{e_3}$ since $e_3 = x_2 e_3$, $\eta_{e_4} = R_{e_4} \nu$ since $e_4 = x_1 e_4 x_2$, and $\{\eta_{e_1}, \eta_{e_2}\}$ is a basis for the \mathfrak{R}_2 -wandering subspace of $\{\nu\}^\perp \cap \mathcal{H}_1$, where $\mathcal{H}_1 = \text{span}\{\xi_w : w \in W_1\}$ is identified with the Fock space for \mathfrak{L}_2 and \mathfrak{R}_2 , and W_1 the set of words in e_1, e_2 . Thus, as in the theorem we have $\mathcal{M} = \sum_{i=1}^5 \oplus \mathfrak{R}_G[\eta_{e_i}]$.

The basis \mathcal{B} in the proof will form an orthonormal basis for the wandering space of $\{\xi_1\}^\perp \cap \{\xi_2\}^\perp$, which is $\text{span}\{\xi_e : e \in E(G)\}$. From the construction outlined in the statement of the theorem, this basis will also have each of its vectors fully supported on some P_j . A gauge unitary \tilde{U} of the type used in the proof will be determined here by a unitary $U \in \mathcal{U}_4$ which fixes ξ_{e_3} and ξ_{e_4} and is allowed to scramble the subspace $\text{span}\{\xi_{e_1}, \xi_{e_2}\}$.

10. PARTLY FREE ALGEBRAS

We now determine in graph-theoretic terms when an operator algebra \mathfrak{L}_G contains the free semigroup algebra \mathfrak{L}_2 as a subalgebra. More generally, let us say that a WOT-closed operator algebra \mathfrak{A} is *partly free* if it contains the free semigroup algebra \mathfrak{L}_2 as a subalgebra in the sense of the following definition.

Definition 10.1. A WOT-closed algebra \mathfrak{A} is *partly free* if there is an inclusion map $\mathfrak{L}_2 \hookrightarrow \mathfrak{A}$ which is the restriction of an injection between the generated von Neumann algebras. If the map can be chosen to be unital, then \mathfrak{A} is said to be *unitally partly free*.

These notions parallel somewhat the requirement that a C^* -algebra contain \mathcal{O}_2 , or that a discrete group contain a free group. Theorems 10.5 and 10.6 determine when the algebras \mathfrak{L}_G are partly free and unitally partly free. We first set aside two results which have intrinsic interest.

Lemma 10.2. *Let $\eta \in Q_x \mathcal{H}_G$ be a unit \mathfrak{R}_G -wandering vector. Suppose that $L_\eta L_\eta^* \leq L_\eta^* L_\eta$. Then $\eta = \sum_u a_u \xi_u$ and for each $u \neq xux$, $a_u = 0$. That is, η is supported on basis vectors corresponding to words forming cycles at the vertex x in G .*

Proof. Since η belongs to $Q_x \mathcal{H}_G$, it follows that $L_\eta^* L_\eta = P_x$, whence $\eta = L_\eta \xi_x \in P_x$ by assumption. Further, the non-zero vectors among $L_\eta \xi_u$ form an orthonormal set, hence

$$1 = \sum_u |a_u|^2 = \|\eta\|^2 = \|L_\eta \eta\|^2 = \sum_u |a_u|^2 \|L_\eta \xi_u\|^2.$$

Thus if $a_u \neq 0$ then $\|L_\eta \xi_u\|^2 = 1$. In particular, ξ_u belongs to the initial space of L_η which is $L_\eta^* L_\eta \mathcal{H}_G = P_x \mathcal{H}_G$. Thus $r(u) = x$, but since $\eta = \sum_u a_u \xi_u$ is in $Q_x \mathcal{H}_G$ we also have $s(u) = x$ when $a_u \neq 0$. It follows that $u = xux$ for all u with $a_u \neq 0$. ■

The cycle algebras \mathfrak{L}_{C_n} from Example 6.5 give the motivational subclass of infinite-dimensional algebras which are not partly free.

Lemma 10.3. *The cycle algebras \mathfrak{L}_{C_n} , $1 \leq n < \infty$, do not contain pairs of partial isometries U, V which satisfy condition (iii) of Theorem 10.5.*

Proof. This readily follows from the matrix function theory description of the cycle algebras \mathfrak{L}_{C_n} since a similar fact holds in the algebras $H^\infty \otimes \mathcal{M}_n$ and their direct sums. This in turn follows from elementary Toeplitz operator theory, or from the fact that these algebras possess a natural faithful trace. ■

We now define the graph-theoretic notions we require. Recall that a cycle in a directed graph is *minimal* if it is not a power of another cycle.

Definition 10.4. We say G has the *double-cycle property* if there are distinct minimal cycles $w = xwx$, $w' = xw'x$ over the same vertex x in G . We say G has the *strong double-cycle property* if for every vertex x in G there is a directed path from x to a vertex lying on a double-cycle.

Theorem 10.5. *The following assertions are equivalent for a countable directed graph G with a finite number of vertices.*

- (i) G has the double-cycle property.
- (ii) \mathfrak{L}_G is partly free.
- (iii) There are non-zero partial isometries U, V in \mathfrak{L}_G with

$$U^*U = V^*V, \quad UU^* \leq U^*U, \quad VV^* \leq V^*V, \quad U^*V = 0.$$

Proof. For (i) \Rightarrow (ii), observe that if w, w' are cycles of minimal length at vertex x , and $w \neq w'$, then we may take $U = L_w, V = L_{w'}$ to define an injection of \mathfrak{L}_2 into \mathfrak{L}_G . Since (ii) clearly implies (iii), it remains to establish the implication (iii) \Rightarrow (i).

By Theorem 8.5 we have the initial projection $U^*U = V^*V$ equal to the sum of certain $P_i \equiv P_{x_i}$. Without loss of generality let us assume

$$(6) \quad U^*U = V^*V = P_1 + \dots + P_k.$$

We establish (i) by induction on k .

For $k = 1$, observe that Lemma 10.2 gives a double-cycle over x_1 when $U^*U = V^*V = P_1$. Indeed, suppose by way of contradiction, that G fails to have the double-cycle property. As $k = 1$, Theorem 8.5 gives $U = L_\eta$ and $V = L_{\eta'}$. (Observe that there is at least a single loop edge over x_1 since U and V are non-zero.) By Lemma 10.2 we deduce that for some minimal cycle w (possibly a single loop edge) both η and η' belong to the subspace

$$\mathcal{H}_w = \text{span}\{\xi_x, \xi_{w^m} : m = 1, 2, \dots\}.$$

But \mathcal{H}_w can be identified with H^2 , and L_w is then identified with the unilateral shift on H^2 . Consider the subspaces

$$H_\eta = \text{span}\{R_w^m \eta : m \geq 0\} \quad \text{and} \quad H_{\eta'} = \text{span}\{R_w^m \eta' : m \geq 0\}.$$

Since these are non-zero invariant subspaces for the multiplicity-one unilateral shift $L_w|_{\mathcal{H}_w}$, it follows from the classical Beurling theorem for H^2 that they have non-empty intersection. This contradicts the hypothesis, since $\mathcal{H}_\eta \subseteq \text{Ran}(L_\eta)$ and $\mathcal{H}_{\eta'} \subseteq \text{Ran}(L_{\eta'})$.

Let $k \geq 2$ and assume (iii) \Rightarrow (i) holds for $m = 1, \dots, k-1$; that is, G contains a double-cycle whenever \mathfrak{L}_G contains a U, V satisfying

(iii) for which $U^*U = V^*V$ is a sum of at most $k - 1$ projections P_i . Let $\mathcal{S} = \{x_1, \dots, x_k\}$ be the vertices corresponding to the projections P_i in (6). We may assume that every vertex $x \in \mathcal{S}$ has the property that a directed path in G leaves it for another vertex in \mathcal{S} . For if $x \in \mathcal{S}$ was a vertex without this property, then $UP_x = P_xUP_x$ and $VP_x = P_xVP_x$ would be non-zero partial isometries in \mathfrak{L}_G with pairwise orthogonal ranges and initial projection P_x containing their final projections. Thus, by the $k = 1$ case, G would contain a double-cycle.

Now fix $x \in \mathcal{S}$ for the moment and consider a directed path w in G that has initial vertex x and final vertex in \mathcal{S} , and passes through a maximal number of vertices in \mathcal{S} without going through the same vertex in \mathcal{S} twice. Let $y \in \mathcal{S}$ be the final vertex of $w = ywx$. We know there is a path from y to another vertex z in \mathcal{S} , but by maximality w must pass through z . Consequently, there is a subset $\mathcal{A} \subseteq \mathcal{S}$ of vertices which lie on a cycle. Let us assume this cycle does not cross itself, and further assume there are no paths in G outside the cycle which have both initial and final vertices belonging to the set of vertices which form the cycle (otherwise G would clearly contain a double-cycle).

Then \mathcal{A} is a proper subset of \mathcal{S} . To see this, suppose $\mathcal{A} = \mathcal{S}$, and let P be the projection which is the sum of all P_x for which x is a vertex on the cycle. Then by the assumptions on \mathcal{A} in the previous paragraph, the algebra $P\mathfrak{L}_GP$ will consist of operators in \mathfrak{L}_G which have non-zero Fourier coefficients only for basis vectors corresponding to words whose letters are edges in the cycle. Let $P_0 = \sum_{x \in \mathcal{A}} P_x \leq P$. Then evidently $P\mathfrak{L}_G|_{P\mathcal{H}_G}$ is unitarily equivalent to \mathfrak{L}_{C_n} for some n , and $PU|_{P\mathcal{H}_G} = P_0UP_0|_{P\mathcal{H}_G} = U|_{P\mathcal{H}_G}$, $PV|_{P\mathcal{H}_G} = P_0VP_0|_{P\mathcal{H}_G} = V|_{P\mathcal{H}_G}$ would yield a pair of partial isometries in \mathfrak{L}_{C_n} satisfying condition (iii). But this cannot happen by Lemma 10.3. Thus \mathcal{A} must in fact be a proper subset of \mathcal{S} .

Let $\mathcal{B}, \mathcal{C}, \mathcal{D}$ be the subsets of \mathcal{S} which make up the complement of \mathcal{A} consisting of respectively: final vertices of paths with initial vertices in \mathcal{A} ; vertices for which there is a path that leaves it and ends at a vertex in \mathcal{A} ; and vertices in \mathcal{S} for which there are no paths to or from vertices in \mathcal{A} . Thus $\mathcal{S} \setminus \mathcal{A} = \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$. We can assume that \mathcal{B} is non-empty. For otherwise, there would be no edges which emerge from the cycle graph of \mathcal{A} and the above reduction argument can be applied, together with the fact that the cycle algebras are not partly free, to view U and V as elements of \mathfrak{L}_H where H is the graph obtained when the sink vertex set \mathcal{B} is removed. But from the definition of \mathcal{D} , there are no directed paths from a vertex in \mathcal{B} to a vertex in \mathcal{D} . Further, there are no paths from \mathcal{B} to \mathcal{C} by the assumptions on \mathcal{A} . Thus there are no paths in G from

\mathcal{B} to any of \mathcal{A} , \mathcal{C} , or \mathcal{D} . Let $P = \sum_{x \in \mathcal{B}} P_x$ be the sum of projections corresponding to vertices in \mathcal{B} . Then $UP = PUP$, $VP = PVP$ are non-zero and P is the sum of strictly less than k of the P_x 's. Hence by induction G has a double-cycle.

Therefore we conclude that G does indeed contain a double-cycle when condition (iii) holds, and this completes the proof. \blacksquare

We next establish the unital version of the previous theorem.

Theorem 10.6. *The following assertions are equivalent for a countable directed graph G with a finite number of vertices.*

- (i) G has the strong double-cycle property.
- (ii) \mathfrak{L}_G is unittally partly free.
- (iii) There are isometries U, V in \mathfrak{L}_G with

$$U^*V = 0.$$

Proof. Condition (ii) clearly implies (iii). For (iii) \Rightarrow (i), notice that in the proof of Theorem 10.5 we actually showed that from every vertex $x \in V(G)$ with $P_x \leq U^*U = V^*V$, there is a directed path into a double-cycle. Thus, in this case, we may apply this argument to every vertex in G since $U^*U = V^*V = I = \sum_{x \in V(G)} P_x$. In particular, G satisfies the strong double-cycle property when (iii) holds.

We next establish (i) \Rightarrow (iii) and (i) \Rightarrow (ii) together. Thus suppose G satisfies the strong double-cycle property. Fix a double-cycle in G and let \mathcal{B} be the (maximal) collection of all vertices which lie on paths going into or on this double-cycle. Let x be a vertex in \mathcal{B} which belongs to the given double-cycle. Then there are minimal cycles $w_1 = xw_1x \neq w_2 = xw_2x$. Let $\mathbb{F}_2^+(w_1, w_2)$ be the set of all words in the generators w_1, w_2 and consider the subspace

$$\mathcal{H}_{w_1, w_2} = \text{span} \{ \xi_x, \xi_w : w \in \mathbb{F}_2^+(w_1, w_2) \}.$$

Fix a positive integer k such that $2^k \geq 2|\mathcal{B}|$. Amongst the 2^k words of length k in $\mathbb{F}_2^+(w_1, w_2)$, choose a set of cardinality $2|\mathcal{B}|$ and label elements of this set by $\{u_y^{(i)} : y \in \mathcal{B}, i = 1, 2\}$. For every $y \in \mathcal{B}$ there is a path v_y such that $v_y = xv_yy$. For $y \in \mathcal{B}$ and $i = 1, 2$ let $w_y^{(i)}$ be the path $w_y^{(i)} = u_y^{(i)}v_y$. Observe that each of the partial isometries $L_{w_y^{(i)}}$ has initial projection $L_{w_y^{(i)}}^*L_{w_y^{(i)}} = P_y$. Further, the entire family of operators $\{L_{w_y^{(i)}} : y \in \mathcal{B}, i = 1, 2\}$ have pairwise orthogonal ranges by design. Thus it follows that the operators

$$U_{\mathcal{B}} = \sum_{y \in \mathcal{B}} \oplus L_{w_y^{(1)}} \quad \text{and} \quad V_{\mathcal{B}} = \sum_{y \in \mathcal{B}} \oplus L_{w_y^{(2)}}$$

are partial isometries in \mathfrak{L}_G with mutually orthogonal ranges contained in \mathcal{H}_{w_1, w_2} and initial projections satisfying

$$U_{\mathcal{B}}^* U_{\mathcal{B}} = \sum_{y \in \mathcal{B}} P_y = V_{\mathcal{B}}^* V_{\mathcal{B}}.$$

Now let $\mathcal{B}_1, \dots, \mathcal{B}_d$ be a maximal family of disjoint sets of vertices of G , where each of these sets is obtained in the same manner as the above set \mathcal{B} . (Choose \mathcal{B}_1 as \mathcal{B} was chosen, then obtain \mathcal{B}_2 in a similar manner from $V(G) \setminus \mathcal{B}_1$, et cetera.) Since the strong double-cycle property holds for G , the disjoint union $\cup_i \mathcal{B}_i = V(G)$. Let $U_{\mathcal{B}_1}, \dots, U_{\mathcal{B}_d}$ and $V_{\mathcal{B}_1}, \dots, V_{\mathcal{B}_d}$ be partial isometries obtained as in the construction of the previous paragraph. Then the operators $\{U_{\mathcal{B}_i}, V_{\mathcal{B}_j} : 1 \leq i, j \leq d\}$ have pairwise orthogonal ranges with initial projections satisfying

$$U_{\mathcal{B}_i}^* U_{\mathcal{B}_i} = \sum_{y \in \mathcal{B}_i} P_y = V_{\mathcal{B}_i}^* V_{\mathcal{B}_i} \quad \text{for } 1 \leq i \leq d.$$

Therefore it follows that the operators $U = \sum_{i=1}^d \oplus U_{\mathcal{B}_i}$ and $V = \sum_{i=1}^d \oplus V_{\mathcal{B}_i}$ are isometries in \mathfrak{L}_G which have mutually orthogonal ranges, and hence condition (iii) holds. Finally, the map which sends the two generators of \mathfrak{L}_2 to U and V induces an injection of \mathfrak{L}_2 into \mathfrak{L}_G , and (ii) holds. This completes the proof. \blacksquare

Remark 10.7. In the finite graph case it is clear from the proof of Theorem 10.6 that the family of paths which determine the partial isometries L_w in the sums defining U and V can be chosen so that they all have the same length. Hence it follows that G has the strong double-cycle property precisely when the transpose graph G^t satisfies the *entrance condition* from [29] (c.f. Definition 5.8), used as a condition which guarantees the existence of isometries with mutually orthogonal ranges in the commutant. Thus in the finite graph case of Theorem 10.6 we have proved this entrance condition on G^t is actually *equivalent* to the existence of isometries with mutually orthogonal ranges in the commutant.

We finish with a brief discussion of hyper-reflexivity. Given an operator algebra \mathfrak{A} , a measure of the distance to \mathfrak{A} is given by

$$\beta_{\mathfrak{A}}(X) = \sup_{L \in \text{Lat } \mathfrak{A}} \|P_L^\perp X P_L\|,$$

where P_L is the projection onto the subspace L and $\text{Lat } \mathfrak{A}$ is the lattice of invariant subspaces for \mathfrak{A} . Evidently, $\beta_{\mathfrak{A}}(X) \leq \text{dist}(X, \mathfrak{A})$, and the algebra \mathfrak{A} is said to be *hyper-reflexive* if there is a constant C such that $\text{dist}(X, \mathfrak{A}) \leq C\beta_{\mathfrak{A}}(X)$ for all X .

The list of known hyper-reflexive algebras is short, but growing. See [2, 4, 6, 7, 9] for examples appearing in the literature. For the algebras \mathfrak{L}_n , hyper-reflexivity was established by Davidson for $\mathfrak{L}_1 = H^\infty$ [7], and by Davidson and Pitts for the free semigroup algebras $n \geq 2$ [9]. In [4] Bercovici introduced a general method motivated by the \mathfrak{L}_n case, and lowered the upper bound for the \mathfrak{L}_n distant constant. In particular, he proved that an algebra is hyper-reflexive with distant constant no greater than 3 whenever its commutant contains a pair of isometries with orthogonal ranges.

Corollary 10.8. *Let G be a countable directed graph with finitely many vertices for which the transpose graph G^t satisfies the strong double-cycle property. Then \mathfrak{L}_G is hyper-reflexive with distant constant at most 3.*

Proof. From Lemma 4.1 the commutant $\mathfrak{L}'_G = \mathfrak{K}_G$ is unitarily equivalent to \mathfrak{L}_{G^t} , which contains a pair of isometries with pairwise orthogonal ranges by the previous theorem. Thus the result follows from a direct application of Bercovici's result. ■

Remark 10.9. Using Corollary 10.8 and separate arguments for graphs without the double cycle property in the transpose graph it can be shown that \mathfrak{L}_G is hyper-reflexive for every finite graph [17]. It would be interesting to have a characterization of general 'hyper-reflexive graphs', although this is likely to be a deep problem.

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REFERENCES

- [1] A. Arias and G. Popescu, *Factorization and reflexivity on Fock spaces*, Int. Equat. Oper. Th. **23** (1995), 268–286.
- [2] W. Arveson, *Interpolation problems in nest algebras*, J. Func. Anal. **20** (1975), 208–233.
- [3] T. Bates, J. Hong, I. Raeburn, W. Szymanski, *The ideal structure of the C^* -algebras of infinite graphs*, e-print arxiv math.OA/0109142, preprint, 2001.
- [4] H. Bercovici, *Hyper-reflexivity and the factorization of linear functionals*, J. Func. Anal. **158** (1998), 242–252.

- [5] A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. **81** (1949), 239–255.
- [6] E. Christensen, *Perturbations of operator algebras II*, Indiana U. Math. J. **26** (1977), 891–904.
- [7] K. Davidson, *The distance to the analytic Toeplitz operators*, Illinois J. Math. **31** (1987), 265–273.
- [8] K. Davidson, E. Katsoulis, D. Pitts, *The structure of free semigroup algebras*, J. reine angew. Math. **533** (2001), 99–125.
- [9] K. Davidson, D. Pitts, *Invariant subspaces and hyper-reflexivity for free semigroup algebras*, Proc. London Math. Soc. **78** (1999), 401–430.
- [10] K. Davidson, D. Pitts, *The algebraic structure of non-commutative analytic Toeplitz algebras*, Math. Ann. **311** (1998), 275–303.
- [11] K. Davidson, *Nest Algebras*, Longman Scientific & Technical, London, 1988.
- [12] L.M. De Alba, J. Peters, *Classification of semicrossed products of finite-dimensional C^* -algebras*, Proc. Amer. Math. Soc. **95** (1985), 557–564.
- [13] R. Douglas, *Banach algebra techniques in operator theory*, Springer-Verlag, New York, 1998.
- [14] B. Duncan, *Universal operator algebras of directed graphs*, preprint, 2004.
- [15] K. Dykema, A. Nica, D. Voiculescu, *Free Random Variables*, C.R.M. Monograph Series, Vol. 1, AMS, 1992.
- [16] K. Hoffman, *Banach spaces of analytic functions*, Dover Publications Inc., New York, 1988.
- [17] F. Jaeck, S.C. Power, *Hyper-reflexivity of free semigroupoid algebras*, preprint, 2003.
- [18] M.T. Jury, D.W. Kribs, *Partially isometric dilations of non-commuting n -tuples of operators*, Proc. Amer. Math. Soc., to appear.
- [19] M.T. Jury, D.W. Kribs, *Ideal structure in free semigroupoid algebras from directed graphs*, J. Operator Theory, to appear.
- [20] E. Katsoulis, D.W. Kribs, *Isomorphisms of algebras associated with directed graphs*, preprint, 2003.
- [21] E. Katsoulis, D.W. Kribs, *Applications of the Wold decomposition to the study of row contractions associated with directed graphs*, Trans. Amer. Math. Soc., to appear.
- [22] D.W. Kribs, S.C. Power, *The reflexive algebras of higher rank graphs*, in preparation, 2004.
- [23] D.W. Kribs, S.C. Power, *Partly free algebras associated with directed graphs*, Operator Theory: Advances and Applications, Birkhauser-Verlag Basel/Switzerland **149** (2004), 381–393.
- [24] D.W. Kribs, *Inductive limit algebras from periodic weighted shifts on Fock space*, New York J. Math. **8** (2002), 145–159.
- [25] D.W. Kribs, *Factoring in non-commutative analytic Toeplitz algebras*, J. Operator Theory **45** (2001), 175–193.
- [26] A. Kumjian, D. Pask, I. Raeburn, *Cuntz-Krieger algebras of directed graphs*, Pacific J. Math **184** (1998), 161–174.
- [27] A. Kumjian, D. Pask, I. Raeburn, J. Renault, *Graphs, Groupoids, and Cuntz-Krieger algebras*, J. Funct. Anal. **144** (1997), 505–541.

- [28] P. Muhly, *A finite dimensional introduction to operator algebra*, A. Katavolos (ed.), Operator Algebras and Applications, 313-354, Kluwer Academic Publishers, 1997.
- [29] P. Muhly, B. Solel, *Tensor algebras, induced representations, and the Wold decomposition*, Can. J. Math. **51** (4), 1999, 850-880.
- [30] G. Popescu, *Multi-analytic operators and some factorization theorems*, Indiana Univ. Math. J. **38** (1989), 693-710.
- [31] G. Popescu, *Multi-analytic operators on Fock spaces*, Math. Ann. **303** (1995), 31-46.
- [32] S. C. Power, *Approximately finitely acting operator algebras*, J. Func. Anal. **189** (2002), 409-469.
- [33] B. Solel, *You can see the arrows in a quiver algebra*, J. Australian Math. Soc., to appear.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF GUELPH,
GUELPH, ONTARIO, CANADA N1G 2W1
E-mail address: dkribs@uoguelph.ca

DEPARTMENT OF MATHEMATICS AND STATISTICS, LANCASTER UNIVERSITY,
LANCASTER, ENGLAND LA1 4YW
E-mail address: s.power@lancaster.ac.uk